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BASIC CONCEPTS IN MECHANICS

compiled by
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PREFACE

The book is compiled from standard text books in mechanics and is tended for use internally in the SEAFDEC Training Department as a reference book for the Marine Engineering Course trainees. The main objective is to help the trainees revise the basic concepts of mechanics which are involved in their everyday engineering work.

Mechanics is based on surprisingly few fundamental principles, but with countless applications. The trainees will therefore be reminded that to obtain the full benefit from mechanics they must learn by understanding the physical and mathematical principles, and never by memorization of the formular, particularly when the emphasis of this book is mostly on engineering problems or, in other words, applied mechanics.

Statics is a part of mechanics which concerns the equilibrium of bodies under the action of forces. This is studied first as a preliminary step to mechanics, and is followed by "Dynamics" which concerns the motion of bodies. A considerable relationship exists between these 2 topics where references are made during the analysis of some problems.

Trainees are expected to gain from an experience this book in solving various practical problems, which are applicable to real situations.

References:

1. Mechanics, by J.L. Meriam, 2nd edition, 1961.
2. Engineering Mechanics, by Lane K. Bransom, 1970.
3. Mechanical Engineer, by David Allan Low, 1970.

PRINCIPLES OF STATICS

1. **Mechanics** is that physical science which deals with the state of rest or motion of bodies under the action of forces. It is the oldest of the physical sciences among which there is no one subject which plays a greater role in engineering analysis than does mechanics. Modern research and development in the fields of vibrations, stability, strength of structures and machine, engine performance, fluid flow, electrical machines and apparatus, and molecular atomic, and sub-atomic behaviour are highly dependent upon the basic principles of mechanics. A thorough understanding of this subject is therefore an absolute prerequisite for work in these and many other fields.

Mechanics is based on surprisingly few fundamental principles which particular emphasis is placed on, together with their application. However, it is essential to develop the ability to represent the work in a clear and logical manner. The tools to gain a wider understanding of the subject and its' application are given in this book in the form of sample problems with solutions and exercises. It is therefore beneficial to find interest & stimulation in many of the real and practical situations included in the problems.

The subject of mechanics is divided into two parts, statics, which concerns the equilibrium of bodies under the action of forces, and dynamics, which concerns the motion of bodies. Dynamics in turn includes kinematics, which is the study of motion of bodies without reference to the forces which cause the motion, and kinetics, which relates the forces and the resulting motions. Theoretical mechanics is primarily the concern of the physicist, while applied or engineering mechanics is of concern mainly to the engineer.

2. **Basic Concepts** before going into detail, certain definitions and concepts which are basic to the study of mechanics should be understood:

Space is a region extending in all directions. Position in space is determined relative to some reference system by linear and angular measurements.

Time is a measure of the succession of events. The unit of time is the second, which is a convenient fraction of the period of the earth's rotation.

Force is the action of one body on another. A force tends to move a body in the direction of its action upon it.

Matter is substance which occupies space.

Inertia is the property of matter causing a resistance to change in motion.

Mass is the quantitative measure of inertia.

A body is matter bound by a closed surface.

A particle is a body of negligible dimensions. In some cases a body of finite size may be treated as a particle, or at other times the particle may be a differential element.

A rigid body is one which exhibits no relative deformation between its parts. This is an ideal hypothesis since all real bodies will change shape to a certain extent when subjected to forces. When such changes are small, the body may be termed rigid without appreciable error. With the exception of deformable springs this book is a treatment of the mechanics of rigid bodies only.

3. **Scalars and Vectors** the quantities dealt with in mechanics are of two kinds. Scalar quantities are those with which a magnitude only is associated. Examples of scalars are time, volume, density, speed, energy and mass. Quantities with which direction as well as magnitude is associated are called vectors. Examples of vectors are displacement, velocity, acceleration, force, moment, and momentum. Scalars may be combined according to the ordinary laws of algebra, whereas combination of vectors requires a particular form of algebra to account for both magnitude and direction.

A vector quantity V is represented by a straight line, Fig.1, having the direction of the vector and having an arrowhead to indicate the sense. The length of the directed line segment represents to some convenient scale the magnitude of V, and the direction of the vector is specified by the angle θ measured from some convenient reference line.

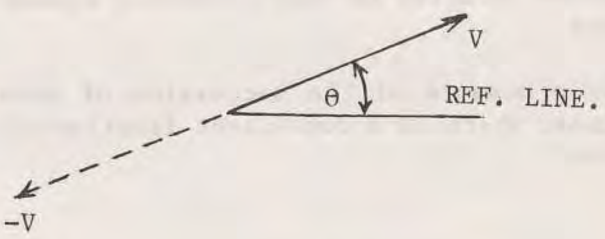


Fig.1

The negative of V is a vector -V directed in the opposite sense to V as shown.

There are 3 kinds of vectors, free, sliding, & fixed. A free vector is one which may be represented by a vector arrow anywhere in space as long as the magnitude and direction remain fixed. If a body moves in a straight line without rotation, then the movement of any point in the body may be taken as a vector, and this vector will describe equally well the motion of every point in the body. Thus the displacement of such a body may be represented by a free vector. Velocity and acceleration are examples also of free vectors.

A sliding vector is one for which a unique line in space must be maintained along which the quantity acts. When dealing with the action of a force on a rigid body, the force may be applied at any point along its line of action without changing its effect on the body as a whole and thus may be considered a sliding vector.

A fixed vector is one for which a unique point of application is specified, and therefore the vector occupies a fixed position in space. The action of a force on a non rigid body must be specified by a fixed vector at the point of application of the force. In this problem the forces and movements internal to the body will be a function of the point of application of the force as well as its line of action.

Vectors may be added and subtracted according to the triangle and parallelogram laws. The two free vectors V_1 and V_2 in Fig.2a may be added head-to-tail to obtain their sum as shown in part (b) of the figure for the triangle law. The order of their combination does not effect their sum. The identical result in maguitude and direction is obtained by completing the parallelogram as shown in part (c) of the figure. In each case this vector addition is expressed

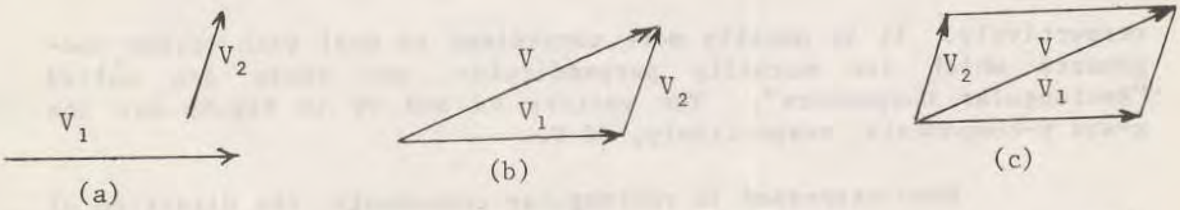


Fig.2

symbolically by the equation $V = V_1 + V_2$, where the symbol $\rightarrow\!\!\!\!+$ is used to denote vector addition in contrast to the + sign used for scalar addition.

The difference V' between the vectors V_1 and V_2 may be obtained by either the triangle or the parallelogram procedure as shown in Fig.3.

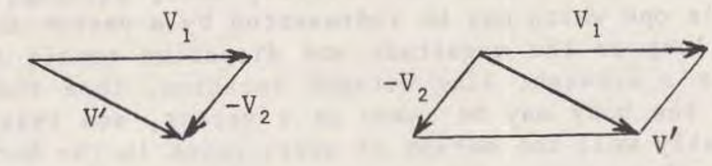


Fig.3

It is necessary only to add the negative of V_2 to V_1 in order to obtain the vector difference. This difference is indicated symbolically by the equation $V' = V_1 \longrightarrow V_2$, where the symbol $\longrightarrow$ is used to denote vector subtraction as distinguished from the - sign used for scalar subtraction.

Any two or more vectors whose sum equals a certain vector V are said to be the components of that vector. Thus the vectors V_1 and V_2 in Fig.4a are the components of V in the directions 1 and 2,

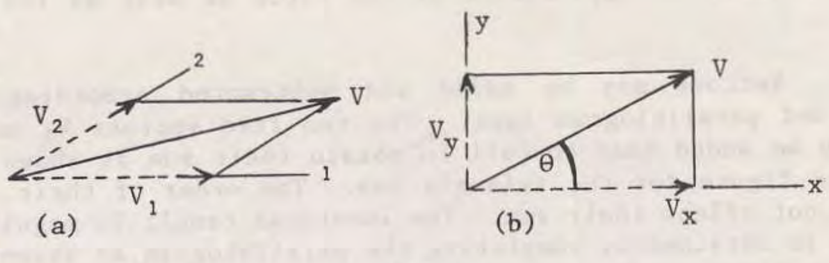


Fig.4

respectively. It is usually more convenient to deal with vector components which are mutually perpendicular, and these are called "Rectangular Components". The vectors Vx and Vy in Fig.4b are the x-and y-components, respectively, of V.

When expressed in rectangular components, the direction of the vector with respect to, say, the x-axis is clearly specified by

$$\theta = \tan^{-1} \frac{Vy}{Vx}$$

4. **Newton's Laws**. Sir Isaac Newton was the first to state correctly the basic laws governing the motion of a particle and to demonstrate their validity. The laws are as follows:

Law I : A particle remains at rest or continues to move in a straight line with a uniform velocity if there is no unbalanced force acting on it.

Law II : The acceleration of a particle is proportional to the resultant force acting on it and is in the direction of this force.

Law III : The force of action and reaction between contacting bodies are equal in magnitude, opposite in direction, and collinear.

Newton's first law explains the natural phenomenon concerning the movement of the Solar System and other Celestial Bodies in the Universe, while the second law forms the basis for most of the analysis in dynamics. As applied to a particle of mass $\underline{m}$ it may be stated as

$$F = ma \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where F is the resultant force acting on the particle and $\underline{a}$ is the resulting acceleration. This equation is a vector equation since the direction of F must be equal to the direction of $\underline{a}$ in addition to the equality in magnitudes of F and $\underline{ma}$. Newton's first law contains the principle of the equilibrium of forces which is the main topic of concern in statics. Actually this law is a consequence of the second law since there is no acceleration when the force is zero, and the particle is either at rest or moves with a constant velocity. The first law adds nothing new to the description of motion but is included since it was a part of Newton's classical statements.

The third law is basic to our understanding of force. It states that forces always occur in pairs of equal and opposite forces. Thus the downward force exerted on the desk by the pencil is accompanied by an upward force of equal magnitude exerted on the pencil by the desk. This principle holds for all forces, variable or constant, regardless of their source and holds at every instant of time during which the forces are applied. Lack of careful attention to this basic law is the cause of frequent error by the beginner.

In addition to formulating the laws of motion for a particle Newton was also responsible for stating the law which governs the mutual attraction between bodies. This law of gravitation is expressed by the equation:

$$F = \gamma \frac{m_1 m_2}{r^2} \qquad \ldots\ldots\ldots\ldots\ldots(2)$$

where F = the mutual force of attraction between two particles.

γ = a universe constant known as the constant of gravitation.

m_1, m_2 = the masses of the two particles.

r = the distance between the centers of the particles.

The mutual forces F obey the law of action and reaction since they are equal and opposite and are directed along the line joining the centers of the particles. Gravitational forces exist between every pair of bodies. On the surface of the earth the only gravitational force of appreciable magnitude is the force due to the earth's attraction. Thus each of two iron spheres 4 inches in diameter is attracted to the earth with a force of 8.90 lb. which is called its weight. On the other hand the force of mutual attraction between them if they are just touching is 0.0000000234 lb. This force is clearly negligible compared with the earth's attraction of 8.90 lb., and consequently the gravitational attraction of the earth is the only gravitational force of any magnitude which need be considered for experiments conducted on the earth's surface.

The weight of a body is the force of attraction of the body to the earth and depends on the position of the body relative to the earth. An object weighing 10 lb. at the earth's surface will weigh 9.99500 lb. at an altitude of 1 mile, 9.803 at an altitude of 40 miles, and 2.50 lb. at an altitude of 4000 miles or a height approximately equal to the radius of the earth. It is at once apparent that the variation in the weight of high-altitude rockets must be accounted for.

Every object which is allowed to fall in a vacuum at a given location on the earth's surface will have the same acceleration $\underline{g}$ as can be seen by combining equations (1) and (2) and cancelling the term representing the mass of the falling object, i.e.,

$$F = m_1 g = \gamma \frac{m_1 \; m_e}{\gamma^2}$$

$$\text{or} \quad g = \gamma \frac{m_e}{\gamma^2}$$

where g = gravitational acceleration

m_1 = mass of the falling body

m_e = mass of the earth

By experiment, value of the constant of gravitation $\gamma = 6.67 \times 10^{-8} \dfrac{cm^3}{gm,\ sec^2}$, where the mass and mean radius of the earth have been found to be 5.98×10^{27} gm. and 6.38×10^8 cm. respectively. Substitute these values to obtain

$$g = (6.67 \times 10^{-8}) \quad \frac{(5.98 \times 10^{27})}{(6.38 \times 10^8)^2} \quad sm/sec^2.$$

$$\text{or} \quad g = 980 \ cm/sec^2.$$

$$= 32.2 \ ft/sec^2.$$

The mass m of a body may be calculated from the results of the simple gravitational experiment. If the gravitational force or weight is W, them, since the body falls with an acceleration g, equation (1) gives:

$$W = mg, \quad \text{or } m = \frac{W}{g} \quad \ldots\ldots\ldots\ldots\ldots\ldots \; (3)$$

5. **Unit.** There are a number of systems of units used in relating force, mass, and acceleration. Engineers use a gravitational system in which length, force, and time are considered fundamental quantities and the units of mass are derived. In English speaking countries, the British or FPS gravitational system is used, these are the units in this book. The derivative unit of mass is called "slug", i.e. $m = \dfrac{F}{a} = \dfrac{lb.}{ft/sec^2} = \dfrac{lb.\ sec^2.}{ft} = slug.$

One slug is therefore the mass of a body which weighs 32.2 lb. at the earth's surface.

Force is in pounds (lb.) or in tons (longton = 2240 lb., short ton = 2000 lb.).

Length in inches, feet, yards, miles (5280 feet)

Time in seconds, minutes, hours.

FORCE SYSTEMS

6. **Force.** A force is defined as the action of one body on another. Being a vector quantity, its effect depends on the direction of the action as well as on the magnitude. Furthermore it is necessary to know where it acts. The action of the cable tension P on the bracket in Fig.5a is represented in Fig.5b by the force vector P.

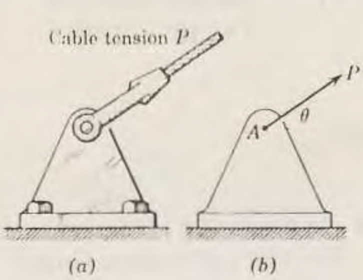


Fig. 5

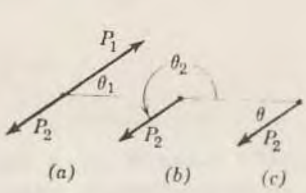


Fig. 6

The effect of this action on the bracket will depend on the magnitude of P, the angle θ, and the location of the point of application A. Changing any one of these three specifications will alter the effect on the bracket. The sense of a force along its line of action is a fourth specification which may be added to distinguish between two forces having - identical lines of action but opposite senses, such as with the forces P_1 and P_2 of Fig.6a. The need for the fourth specification disappears if the obtuse angle θ_2 is used as in Fig.6b. Since directions of lines are most frequently measured and represented by acute angles, the specification of an obtuse angle is often inconvenient. The question is most easily resolved if the direction of the force is indicated by an acute angle such as θ in Fig.6c. As long as the force and angle are clearly represented on a sketch or are adequately described, there is a need for only the three specifications of force: magnitude, direction, and point of application.

Force is applied either by direct mechanical contact or by remote action. Gravitational and electrical forces are the two examples of force applied by remote action. All other actual forces are applied through direct physical contact.

The action of a force on a body can be separated into two effects, external and internal. For the bracket of Fig.5 the effects of P external to the bracket are the reactions or forces exerted on the bracket by the foundation and bolts in consequence of the action of P. Forces external to a body are then of two kinds, applied or active forces and resulting or reactive forces. The effects of P internal to the bracket are the resulting internal movements and forces distributed throughout the material of the bracket. The relation between internal forces and internal movements involves the material properties of the body and is studied in the subjects of strength of materials, elasticity, and plasticity.

In dealing with the mechanics of rigid bodies where concern is given only to the net external effects of forces, experience shows that it is not necessary to restrict the action of an applied force to a given point. Thus the force P acting on the rigid bracket in Fig.7 may be applied at A or at B or at any other point on its action line, and the net external effect of P on the bracket will not change. This situation is described by the "PRINCIPLE OF TRANSMISSIBILITY" which states that a force may be applied at any point on its given line of action without altering the resultant effects of the force external to the rigid body on which it acts. When the resultant external effects only of a force are to be investigated, the force may be treated as a sliding vector, and it is necessary and sufficient to specify the magnitude, direction, and line of action of the force.

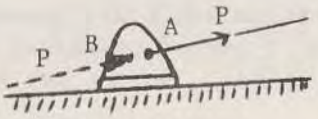


Fig.7

Forces may be either concentrated or distributed. Actually every contact force is applied over a finite area and is therefore distributed. When the dimensions of the area are negligible compared with the other dimensions of the body, the force may be considered as concentrated at a point. Force may be distributed over an area, as in the case of mechanical contact, or it may be distributed over a volume when gravity or magnetic force is acting. The "weight" of a body is the force of gravity distributed over its volume and may be taken as a concentrated force acting through the center of gravity. The position of the center of gravity is usually obvious from consideration of symmetry. If the position is not clear, then a separate calculation, explained later will be necessary to locate the center of gravity.

A force may be measured either by comparison with other known forces, using a mechanical balance, or by the calibrated movement of an elastic spring. All such comparisons or calibrations have as their basis a primary standard. The standard pound for the United States is legally defined as 0.4535924277 times the international kilogram and is that force required to support this portion of the standard kilogram in a vacuum and under the standard conditions at sea level at which the acceleration of gravity is 32.1740 ft./sec.^{2}.

The characteristic of a force expressed by Newton's third law must be carefully observed. The action of a force is always accompanied by an equal and opposite reaction. It is essential to fix clearly in mind which force of the pair is involved. The answer is always clear when the body in question is isolated and the force exerted on that body (not by the body) is represented.

7. **Addition of forces.** Two concurrent forces F1 and F2 may be added by the triangle or parallelogram laws to obtain their sum or resultant R as shown in Fig.8a. If the two forces lie in the same plane but are applied at two different points as in Fig.8b, they may be moved along their lines of action and their vector sum R completed at the point of concurrency. The resultant R may replace F1 and F2 without altering the external effects on the body. The triangle law requires moving the line of action of one of the forces as shown in Fig.8c. In Fig.8d the same two forces are added, and although the correct magnitude and direction of R are preserved, the correct line of action is lost. This type of combination should be avoided. Mathematically the sum of the two forces may be written by the vector equation.

$F_1 \nrightarrow F_2 = R$

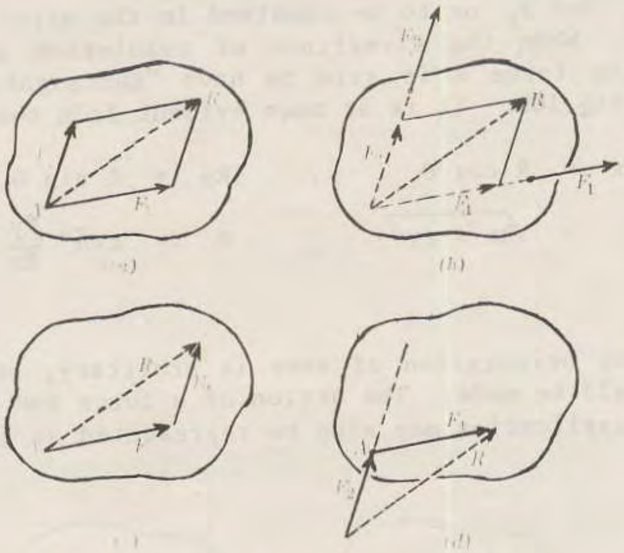


Fig.8

A special case of addition is presented when the two forces F_1 and F_2 are parallel, Fig.9. Graphically they may be combined by first adding two equal, opposite, and collinear forces F of convenience magnitude which together produce no external effect on the body. Adding F_1 and F and combining with the sum of F_2 and F yield the resultant R correct in magnitude, direction, and line of action. The procedure here is also useful in obtaining a graphical combination of two forces which are almost parallel and hence may have a remote point of concurrency.

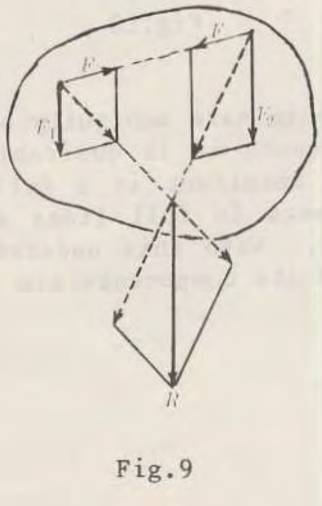


Fig.9

8. **Resolution:** The force R of Fig.10a is said to have the components F_1 and F_2 or to be resolved in the direction 0-1 and 0-2, respectively. When the directions of resolution are mutually perpendicular, the force R is said to have "Rectangular Components" Rx and Ry as in Fig.10b. It is at once evident from the figure that:

$$Rx = R\cos\theta \quad , \quad Ry = R\sin\theta$$

$$R = \sqrt{Rx^2 + ry^2} \quad , \quad \theta = \tan^{-1}\frac{Ry}{Rx} \quad \ldots\ldots\ldots\ldots(4)$$

The orientation of axes is arbitrary, and a selection as in Fig.10c could be made. The action of a force and its components at the point of application may also be represented as in Fig.10d.

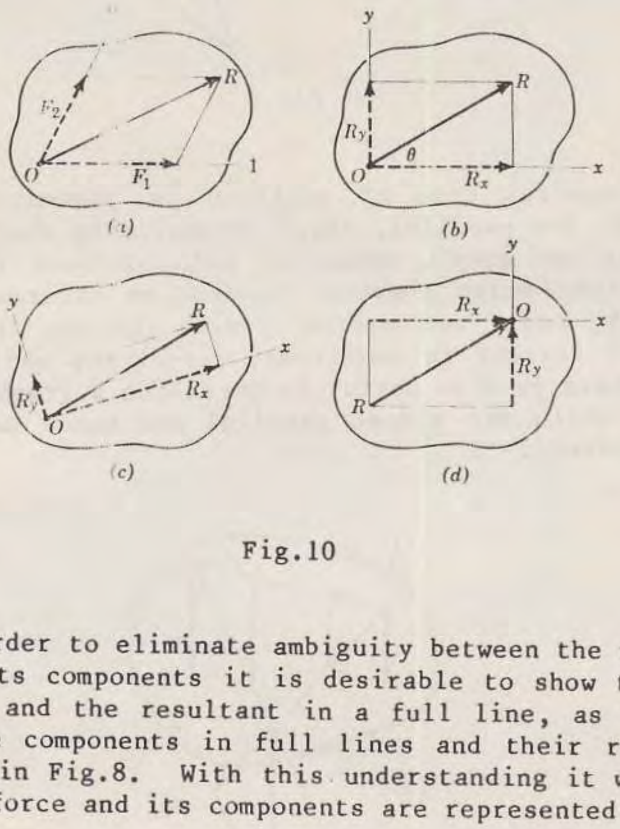


Fig.10

In order to eliminate ambiguity between the representation of a force and its components it is desirable to show the components in dotted lines and the resultant in a full line, as in Fig.10, or else to show the components in full lines and their resultant in a dotted line, as in Fig.8. With this understanding it will always be apparent that a force and its components are represented and not three separate forces.

Many problems are three dimensional, and it becomes necessary to consider the 3 mutually perpendicular components of a force. In Fig.11 force F is resolved into its rectangular components Fx, Fy, Fz by forming the edges of the rectangular parallellepiped of which F is a diagonal.

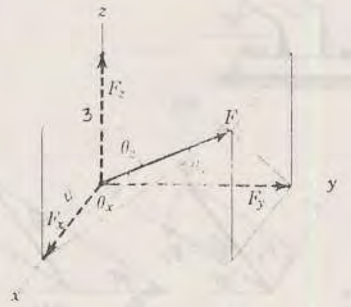


Fig.11

When the angles made by F with the x, y, and z axes are designated by θx, θy, θz respectively, it is evident from the right triangles involved that

$$F_x = F \cos \theta_x$$

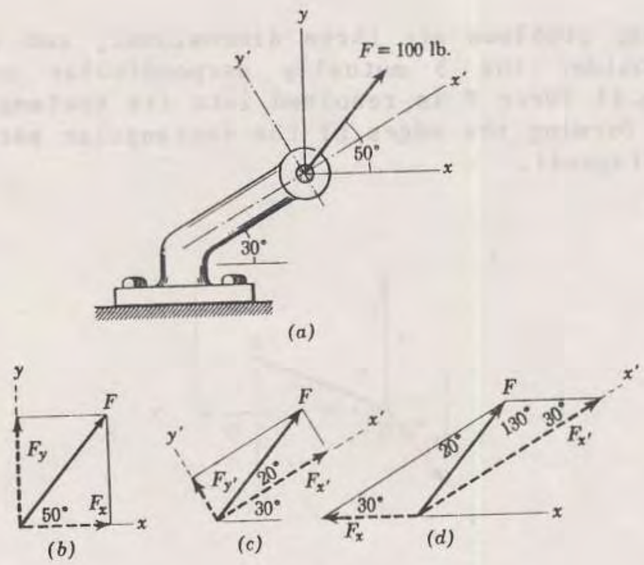
$$F_y = F \cos \theta_y \qquad \text{and } F = \sqrt{F_x^2 + F_y^2 + F_z^2} \;\ldots\ldots\ldots \; (5)$$

$$F_z = F \cos \theta_z$$

The direction cosines of A are cos θx, Cos θy, Cos θz. The choice of the orientation of the axes is quite arbitrary, and considerations of convenience usually determine this selection.

SAMPLE PROBLEMS

1. The 100 lb. force is applied to the bracket as shown in part (a) of the figure. Determine the rectangular components of F in (1) the x and y directions and (2) the x′ and y′ directions. Also (3) find the components of F in the x and x′ directions.



PROB. 1

Solution. Part (1) The x and y components of F are shown in part (b) of the figure and are

$$F = F\cos\Theta x = 100\cos 50^\circ = 64.3 \text{ lb.}$$

$$F = F\sin\Theta x = 100\sin 50^\circ = 76.6 \text{ lb.} \qquad \underline{\text{Ans}}$$

Part (2) The x´ and y´ components of F are shown in part (c) of the figure and are

$$Fx' = F\cos\Theta x' = 100\cos 20^\circ = 94.0 \text{ lb.}$$

$$Fy' = F\sin\Theta x' = 100\sin 20^\circ = 34.2 \text{ lb.} \qquad \underline{\text{Ans}}$$

Part (3) The components of F in x and x´ directions are obtained by completing the parallelogram as indicated in part (d) of the figure. Fx and Fx´ in the directions indicated are obtained from the law of sines. Thus

$$\frac{Fx}{\sin 20^\circ} = \frac{F}{\sin 30^\circ}; \quad Fx = \frac{0.342}{0.500} \times 100 = 68.4 \text{ lb.},$$

$$\frac{Fx}{\sin 130^\circ} = \frac{F}{\sin 30^\circ}; \quad Fx = \frac{0.766}{0.500} \times 100 = 153 \text{ lb.} \qquad \underline{\text{Ans.}}$$

2. If the force F in Fig.11 is 100 lb. and passes through a point whose x, y, and z coordinates are 4, 5, and 3 respectively, determine the rectangular components of F. Also find the component of F in the x-y plane.

Solution. The direction cosines are

$$\cos \theta_x = \frac{4}{\sqrt{4^2 + 5^2 + 3^2}} = 0.566$$

$$\cos \theta_y = \frac{5}{\sqrt{4^2 + 5^2 + 3^2}} = 0.707$$

$$\cos \theta_z = \frac{3}{\sqrt{4^2 + 5^2 + 3^2}} = 0.424$$

The components of F are then

$$Fx = F \cos \theta x = 100 \times 0.566 = 56.6 \text{ lb.},$$

$$Fy = F \cos \theta y = 100 \times 0.707 = 70.7 \text{ lb.}, \qquad \underline{\text{Ans.}}$$

$$Fz = F \cos \theta z = 100 \times 0.424 = 42.4 \text{ lb.}$$

The cosine of the angle θxy made by F with the x-y plane is

$$\cos \theta xy = \frac{\sqrt{4^2 + 5^2}}{\sqrt{4^2 + 5^2 + 3^2}} = 0.906,$$

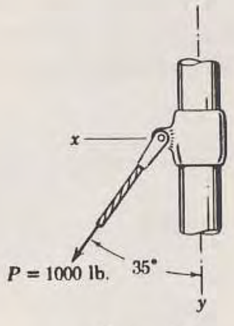
and the component of F in the x-y plane is

$$Fxy = F \cos \theta xy = 100 \times 0.906 = 90.6 \text{ lb.} \qquad \underline{\text{Ans.}}$$

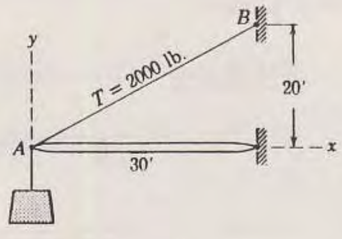
PROBLEMS

3. The cable exerts a pull of P = 1000 lb. on the mast band. How much of this force is exerted on the fitting in the x and y directions?

<u>Ans</u>. Px = 574 lb.,
Py = 819 lb.



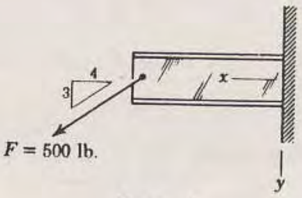
PROB. 3



PROB. 4

4. Determine the x and y components of the force exerted on the pin at A by the 2000 lb. cable tension T. <u>Ans</u>. Tx = 1660 lb., Ty = 1110 lb.

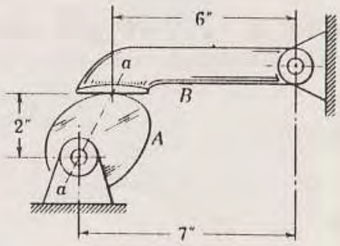
5. Find the x and y components of the force F



PROB. 5

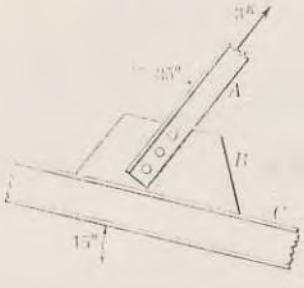
6. A force F which acts in the x-y plane has a magnitude of 200 lb. and a direction $\theta x = 130°$ Find the x and y components of F. <u>Ans</u>. Fx = -128 lb., Fy = 153 lb.

7. The force F between the cam A and the follower B is normal to their surfaces of contact and equals 150 lb. for the position shown. Determine the component of F along a-a in the direction of the cam bearing.

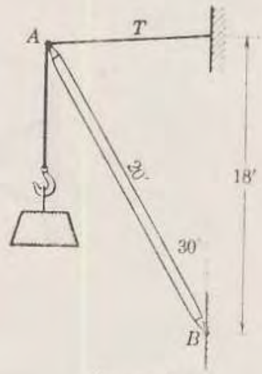


PROB. 7

8. The structural member A is under a tensile load of 3 kips (1 kip = 1000 lb.). Determine the amount of force F transmitted by this member to the gusset plate B in the direction of the I-beam C.
<u>Ans</u>. F = 1.03 kips.



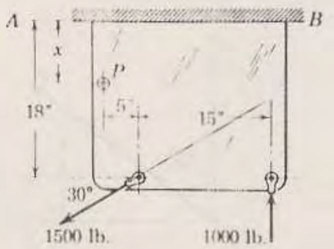
Prob. 8



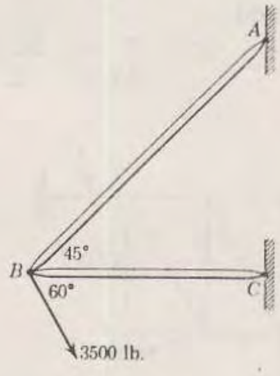
Prob. 9

9. The tension T in the supporting cable of the 20 ft. boom is 2000 lb. Resolve T into two forces applied at A: one, Fn normal to the boom and the other, Ft, along the boom.

10. The two forces shown are to be replaced by an equivalent force R applied at the point P. Locate P by finding its distance x from AB, and specify the magnitude of R and the angle θ it makes with the horizon. <u>Ans</u>. x = 5.49 in., R = 1323 lb., θ = 10° 54'



Prob. 10

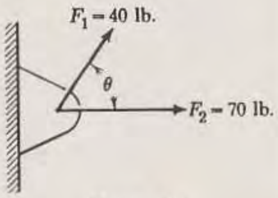


Prob. 11

11. In finding the forces exerted on the pin connections at A and C the 3500 lb. force is resolved into two forces, one along the line AB and the other along BC. Find these components.

12. At what angle θ must F_1 be applied in order that the resultant R of F_1 and F_2 be equal to 100 lb.? For this condition what will be the angle β between R and the horizontal?

Ans. $\theta = 51°\ 19'$
$\beta = 18°\ 12'$



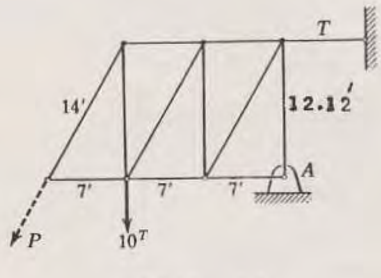
PROB. 12

13. A 100 lb. force which makes an angle of 45° with the horizontal x-axis is to be replaced by two forces, a horizontal force F and a second force of 75 lb. magnitude. Find F.

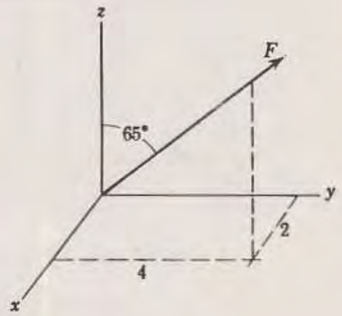
Ans. F = 95.7 lb. or F = 45.7 lb.

14. If the weight of the truss is neglected, the pin force acting on the truss at A is equal and opposite to the resultant of the 10 ton load and the corresponding cable tension T acting on the truss. If the 10 ton load is removed and a force P is applied as indicated by the dotted line, determine the magnitude of P and the corresponding increase ΔT in cable tension so that the reaction on the pin A will remain unchanged.

Ans. P = 11.55 tons,
ΔT = 5.77 tons.



PROB. 14



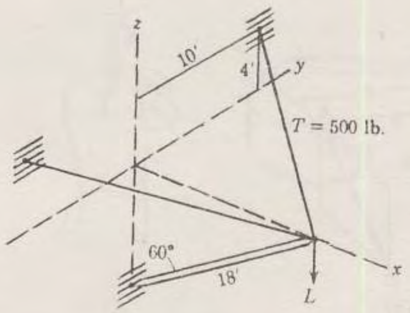
PROB. 15

15. If the x-component of the force F is 300 lb., find F.

Ans. F = 740 lb.

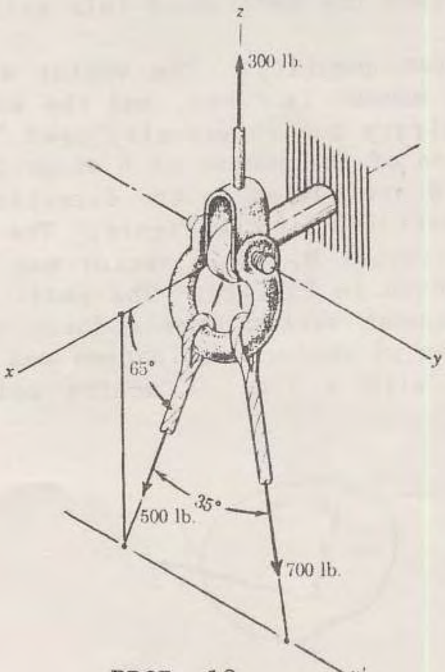
16. Determine the direction cosines of the force F of problem 15.

17. The tension T in one of the two supporting cables for the boom is 500 lb. Resolve the force exerted by this cable on the boom into its x, y, and z components. Ans. $T_x = -411$ lb., $T_y = 264$ lb., $T_z = 106$ lb.



PROB. 17

18. The fixed eyebolt is subjected to the three forces shown. If a single force is to replace the three given forces without altering the effect on the eyebolt, find the correct value of the force F and its direction cosines. Ans. F = 906 lb., $\cos\theta_x = 0.501$, $\cos\theta_y = 0.444$, $\cos\theta_z = -0.743$



PROB. 18

9. **Moment:** The tendency of a force to rotate the body upon which it acts about a certain axis is know as the moment of the force about that axis. In Fig.12a consider the moment M about the axis 0-0 of a force R applied at any point A on the body as shown. This moment

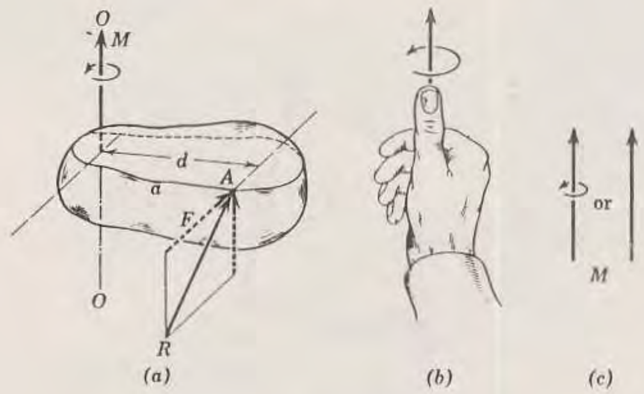


Fig.12

is due entirely to the component of R in a plane normal to the axis and equals this component F multiplied by the perpendicular distance or moment arm d from the line of action of F to the axis 0-0. Thus

$$M = F.d \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

The component of R normal to F is parallel to 0-0 and hence exerts no tendency to rotate the body about this axis.

Moment is a vector quantity. The vector direction is along the axis about which the moment is taken, and the sense of the vector is specified by the arbitrary but universally used "Right Hand Rule". To represent the direction of the moment of R about 0-0 in Fig.12a the fingers of the right hard are curled in the direction of the tendency to rotate as shown in part (b) of the figure. The thumb then points in the direction of the vector M. This vector may be represented in either of the two ways shown in Fig.12c. The small curl is sometimes used to distinguish a moment vector from a force vector. A moment vector obeys all the rules of vector combination and may be considered to be a sliding vector with a line of action coinciding with the moment axis.

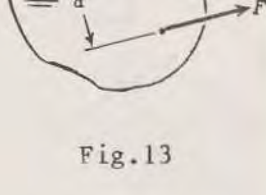


Fig.13

When dealing with forces all of which act in a given plane it is customary to speak of the moment about a point. Actually the moment with respect to an axis normal to the plane and passing to the point is implied. Thus the moment of force F about point O in Fig.13 is Mo = F.d and is counterclockwise. Vector representation of moments for coplanar forces is not convenient since the vectors are either out from the paper (counterclockwise) or into the paper (clockwise). Since the addition of parallel free vectors may be accomplished with scalar algebra, the moment directions may be accounted for by using a plus sign (+) for counterclockwise moments and a minus sign (-) for clockwise moments, or vice versa. It is necessary only to be consistent within a given problem in using either sign convention.

One of the most important principles of mechanics is "Varignon's Theorem", or the principle of moments, which states that the moment of a force about any point is equal to the sum of the moments of its components about the same point. To prove this statement consider a force R and two components P and Q acting at point A, Fig.14. Point O is selected arbitrarily as the moment center. Construct the line AO and project the three vectors onto the normal to

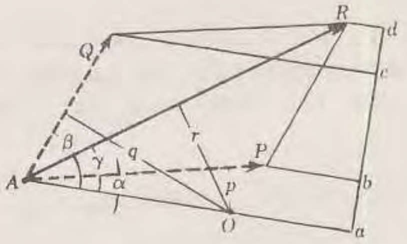


Fig.14

this line. Also construct the moment arms p, q, r, of the three forces to point O and designate the angles of the vectors to the line AO by α, β, γ as shown in the figure. Since the parallelogram whose sides are P and Q requires that ac = bd, it is evident that

$$ad = ab + bd = ab + ac$$

$$\text{Or } R \sin\gamma = P \sin\alpha + Q \sin\beta$$

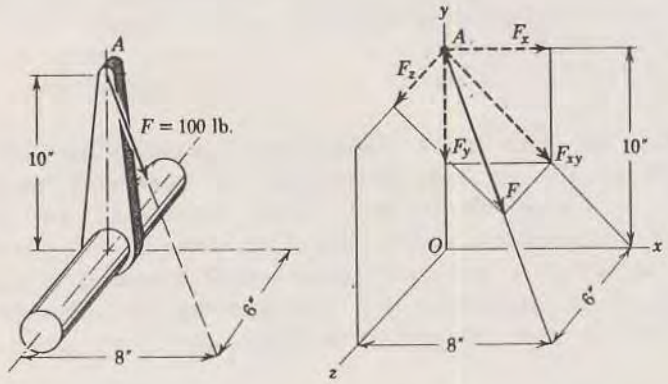
Multiplying by the distance AO and substituting the values of p, q, r give

$$Rr = Pp + Qq,$$

which proves that the moment of a force about any point equals the sum of the moments of its two components about the same point. Varignon's theorem need not be restricted to the case of only two components but applies equally well to three or more since it is always possible by direct combination to reduce the number of components to two for which the theorem was proved. The theorem may also be applied to the moments of other fixed or sliding vectors.

SAMPLE PROBLEM

1. A force F = 100 lb., is applied to the arm shown, which is attached to the rigid shaft. Determine the moment M of F about the shaft axis.



Solution: In the right hand part of the figure the force F is shown resolved into components Fxy in the x-y plane which is normal to the shaft axis and the component Fz. The moment M of F about the shaft or z-axis is

$$M = Fxy.d,$$

where d is the perpendicular distance from Fxy to O. The cosine of the angle between F and Fxy is $\sqrt{8^2 + 10^2} / \sqrt{8^2 + 6^2 + 10^2} =$ 0.906, and therefore Fxy = 100 x 0.906 = 90.6 lb.

The moment arm d equals OA multiplied by the sine of the angle between Fxy and OA, or $d = 10 \times \frac{8}{8^2 + 10^2} = 6.25$ in.

Hence the moment of F about the z-axis is

$$M = 90.6 \times 6.25 = 566 \text{ lb. in} \qquad \underline{\text{Ans.}}$$

Calculation of the moment is somewhat simplified by resolving Fxy into Fx and Fy. It is clear that Fy exerts no moment about the Z-axis so that Fx only need be considered. The direction cosine of F with respect to the x-axis is $8\sqrt{8^2 + 6^6 + 10^2} = 0.566$ so that Fx = 100 x 0.566 = 56.6 lb. Thus

$$M = 56.6 \times 10 = 566 \text{ lb. in}$$

Expressed as a vector, M would be in the negative Z-direction.

PROBLEMS

1. Prove that the moment of a force about a point equals twice the area of the triangle defined by that point and the two extremities of the force vector.

2. A force F is applied at a point whose coordinates are x, y, and z. Show that the moment of F about the x-axis is Mx = Fz.Y - Fy.Zand write the expressions for the moments My and Mz about the y and z axes, respectively.

3. Determine the moment M of the 50 lb. force about the point A. Ans. M = 1160 lb. in

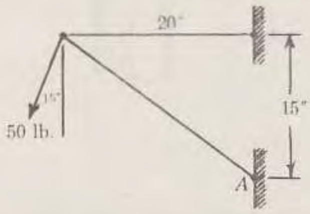


Fig. 3

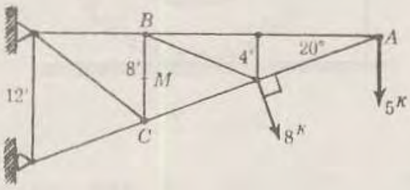


Fig. 4

4. Determine the moment M of the 5 kip force about point M for the overhanging truss. Ans. M = 109,900 lb. ft.

5. The force F of contact between the teeth of the gears is normal to their contact surfaces. If the smaller gear transmits a torque (moment) of 200 lb. in about its shaft O and the pressure angle is 20° as shown, find F.

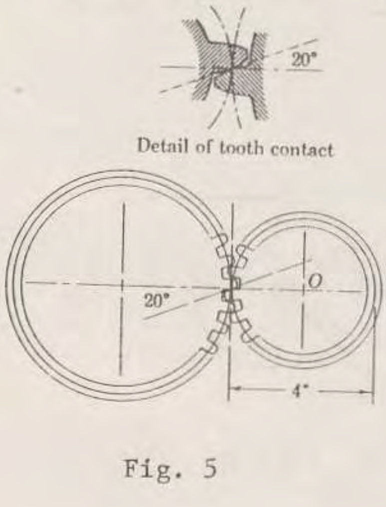


Fig. 5

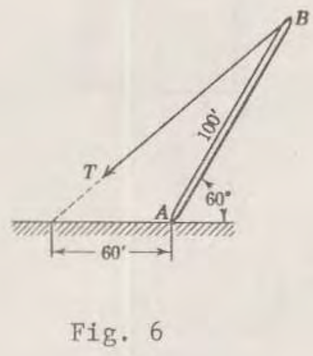


Fig. 6

6. In raising the mast AB from the position shown the tension T in the cable must supply a moment about A of 45,000 lb.ft. Find T. Ans. T = 1212 lb.

7. A force of 40 lb. is applied to the end of the wrench to tighten a flange bolt which holds the wheel to the axle. Determine the moment M produced by this force about the center O of the wheel for the position of the wrench shown.

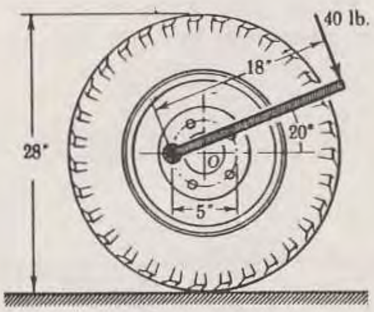


Fig. 7

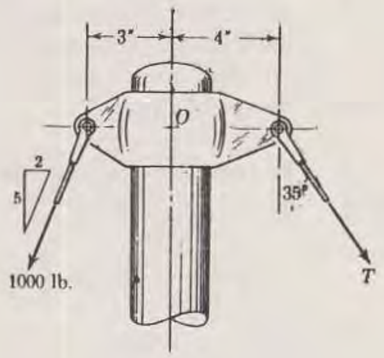


Fig. 8

8. The top of the mast is supported by the two cables attached to the mast fitting as shown. In order to prevent bending of the mast at the fitting there must be no resulting moment about point O on the center line of the mast. Determine the necessary value of T. (Will this value of T eliminate bending of the mast at points below O?)

9. The single-cylinder gasoline engine represented in the figure has a stroke of 8 in., and the connecting rod is 14 in. between bearing centers. In the position shown the rod is under compression (in the direction of the rod) of 4500 lb. Determine the moment M of this force about the crankshaft axis. Ans. M = 17,330 lb.in.

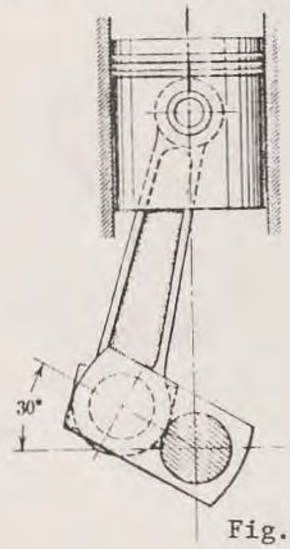


Fig. 9

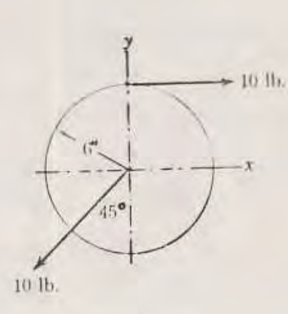


Fig. 10

10. A wheel is subjected to two forces as shown in the figure. Determine graphically the coordinates of the point on the rim of the wheel about which the combined moment M of the two forces is a maximum. Find M for this point. Ans. x = -5.5 in., y = -2.3 in., M = 106 lb.in

11. Determine the moment Mx about the shaft axis of the 500 lb. force applied to the bracket shown.

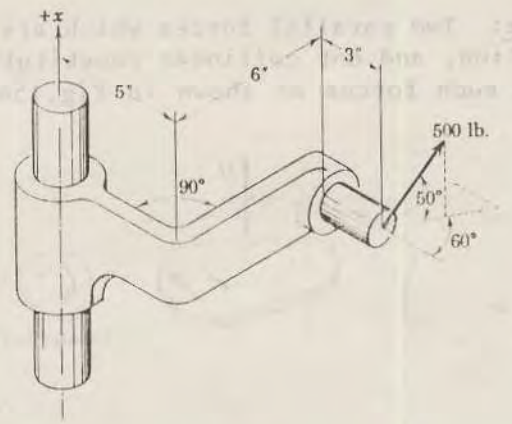


Fig.11

12. The steel bracket is fastened to the surface A by a bolt B whose axis is normal to this surface. Determine the moment M of the 80 lb. force about the bolt axis. Ans. M = 992 lb.in

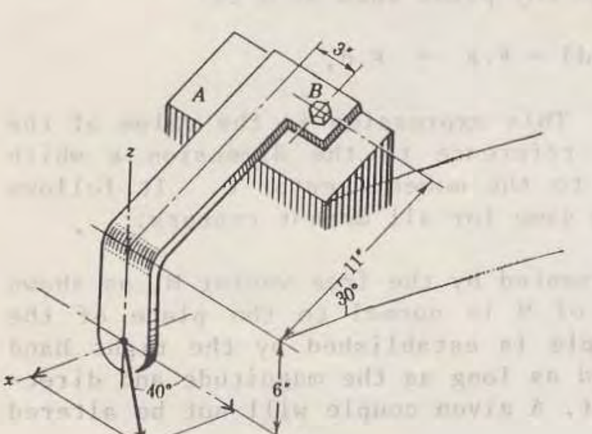


Fig. 12

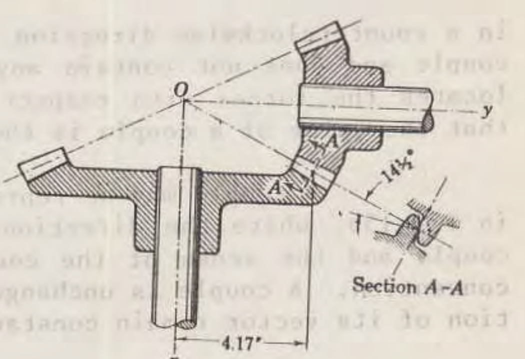


Fig. 13

13. Contact between the two bevel gears shown, which have a speed ratio of 2:1, may be assumed to occur at a point 4.17 in. from the axis of the larger gear. If the contact force is normal to the tooth surfaces shown in the auxiliary view and is 100 lb., determine the magnitude of its moment M about each gear axis.

Ans. Mx = 404 lb.in., My = 202 lb.in.

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10. **Couple:** Two parallel forces which are equal in magnitude, opposite in direction, and not collinear constitute a couple. Consider the action of two such forces as shown in Fig.15a. These two forces

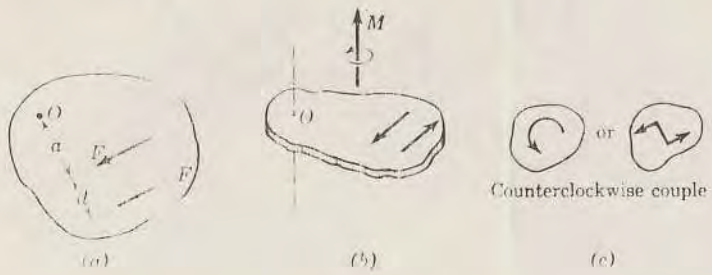


Fig.15

cannot be combined into a single force since their sum in every direction is zero. Their effect is entirely one of producing a tendency of rotation. The combined moment of the two forces about an axis normal to their plane and passing through any point such as O is:

$$M = F\ (a+d) - F.a = F.d,$$

in a counterclockwise direction. This expression is the value of the couple and does not contain any reference to the dimension a which locates the forces with respect to the moment center O. It follows that the value of a couple is the same for all moment centers.

A couple may be represented by the free vector M, as shown in Fig.15b, where the direction of M is normal to the plane of the couple and the sense of the couple is established by the right hand convention. A couple is unchanged as long as the magnitude and direction of its vector remain constant. A given couple will not be altered

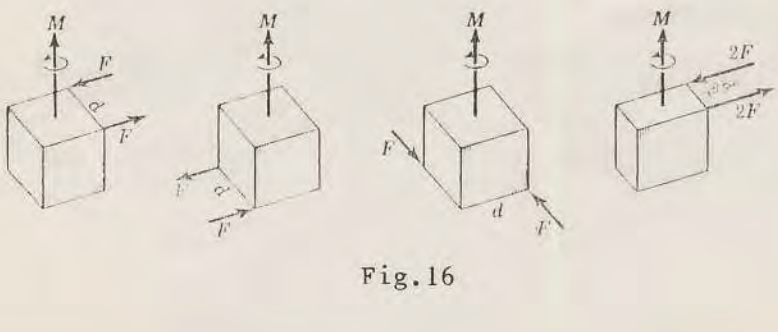


Fig.16

by changing the values of F and d as long as their product remains the same. Likewise a couple is not effected by allowing the forces to act in any one of the parallel planes. In Fig.16 are shown four different configurations of the same couple M = F.d. When dealing with couples due to forces all of which act in the same or parallel planes, the couple vectors will be perpendicular to the plane or planes. In this case it is more convenient to represent such a couple by either of the conventions shown in Fig.15c, where the counterclockwise couple may be taken as positive and a clockwise couple negative or vice versa.

Couples which act in non-parallel planes may be added by the ordinary rules of vector combination. Thus the two couples M and M_2 in Fig.17a due to forces which act in the two planes indicated may be replaced by their vector sum M shown in Fig.17b., which represents a couple due to forces in a plane normal to M.

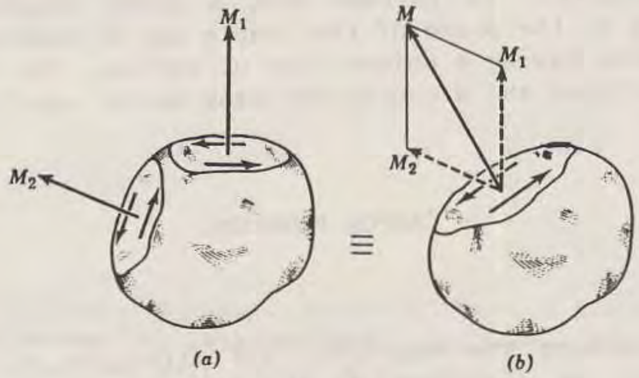


Fig.17

11. **Resolution of a Force into a Force and a Couple:** The effect of a force on a body is, in general, twofold. It tends to push or pull the body in the direction of the force, and it tends to rotate the body about any axis not intersecting its line of action or parallel to it. Analysis of this dual effect is often facilitated by replacing the force by an equal force applied along a different but parallel line of action and by a corresponding couple to compensate for the change in the moment of the force due to moving it to the new position. Consider a body with a force F acting at point A as shown in Fig.18a. At any other point B two equal and opposite forces F may be applied as in Fig.18b. with no external effect on the body. If these forces are parallel to the original force F, then a couple M = F.d is formed by the original F and the force F in the opposite direction at B. In representing the couple by the curled arrow in Fig.18c, it becomes evident that the original force F has been replaced by or resolved into a force of the same magnitude and direction as F, but having a different line of action, and a couple. The magnitude of the

couple equals the value of the force multiplied by the distance through which its line of action has been shifted. The direction of the couple is the same as the direction of the moment of the original force about a point on its new line of action. A force may always be

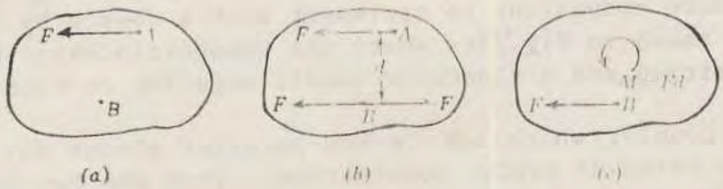
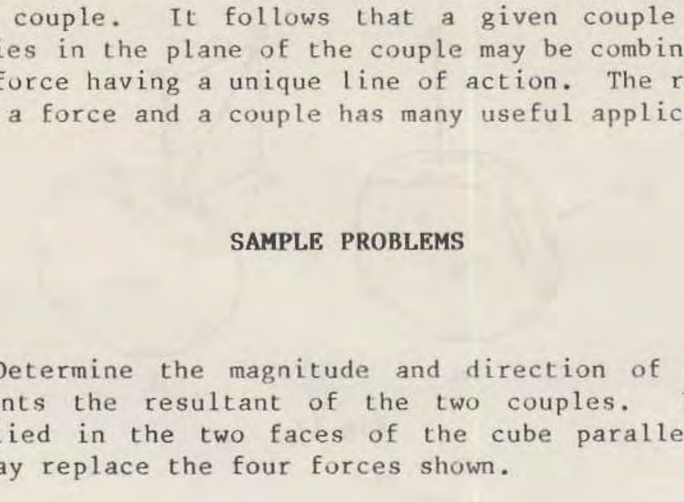


Fig.18

replaced by an equal force having any parallel line of action and the corresponding couple. It follows that a given couple and a given force which lies in the plane of the couple may be combined to yield a single equal force having a unique line of action. The representation of a force by a force and a couple has many useful applications.

SAMPLE PROBLEMS

1. Determine the magnitude and direction of the vector M which represents the resultant of the two couples. Find the two forces F applied in the two faces of the cube parallel to the x-y plane which may replace the four forces shown.



Solution: The couple due to the 10 lb. forces is M = 10 x 2 = 20 lb.in, and the vector is in the y-direction by the right hand convention. The couple due to the 3 lb. forces is M_2 = 3 x 4 = 12 lb.in., and the vector is in the x-direction. The vector sum is

$$M = \sqrt{M_1^{\ 2} + M_2^{\ 2}} = \sqrt{(20)^2 + (12)^2} = 23.3 \text{ lb.in.} \qquad \underline{\text{Ans.}}$$

and the direction of M with the x-axis is given by:

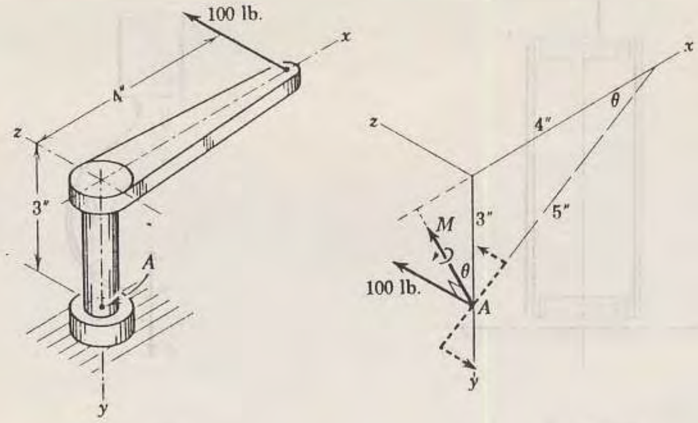
$$\theta = \tan^{-1}\frac{M_1}{M_2} = \tan^{-1}\frac{20}{21} = 59° \ 2' \qquad \underline{\text{Ans.}}$$

The forces F lie in a plane normal to M and hence make an angle of 0 = 59°2' with the y-direction. The magnitude of the forces is:

$$F = \frac{M}{d} = \frac{23.3}{4} = 5.83 \text{ lb.} \qquad \underline{\text{Ans.}}$$

---

2. A force of 100 lb. is applied to a lever which is attached to a fixed shaft as shown. In determining the effect of the force on the shaft at a section such as A it is convenient to replace the force by a force at A and a couple M. Find the magnitude and direction of M.



Solution: Shifting the force a distance of $\sqrt{3^2 + 4^2} = 5$ in. to a parallel position at A requires the addition of a couple of magnitude

$$M = 100 \times 5 = 500 \text{ lb.in.} \qquad \underline{\text{Ans.}}$$

The vector M representing the couple is in the x-y plane normal to the force vector and makes an angle with the y-axis of

$$\theta = \cos^{-1}\frac{4}{5} = 36° \ 52' \qquad \underline{\text{Ans}}.$$

PROBLEMS

1. The action of the 2000 lb. load on the column may be analyzed by considering it to produce a compression along the center line and a couple. If this couple is 6000 lb.in., find d.

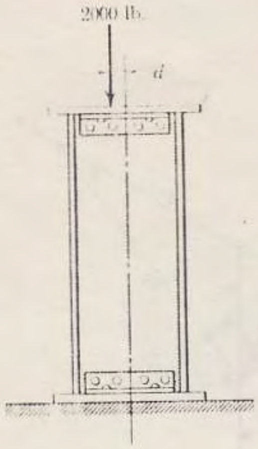


Fig. 1

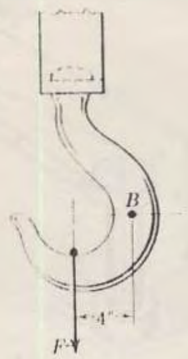


Fig. 2

2. In designing the lifting hook the forces acting on a horizontal section through B may be determined by replacing F by an equivalent force at B and a couple. If the couple is 3 ton-ft., determine F. Ans. F = 18,000 lb.

3. Each propeller of the twin-screw ship shown develops a full speed thrust of 80,000 lb. In manoeuvring the ship one propeller is turning full speed ahead and the other full speed in reverse. What force F must each tug exert on the ship to counteract the turning effect of the ship's propellers?

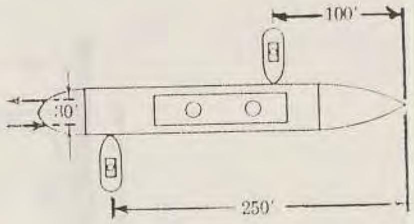


Fig.3

4. The truss shown supports the vertical load of 5000 lb. and is held in place by the reactive forces exerted on it by the supports as indicated in the figure. The force Ay and the applied load constitute a couple, and the remaining two forces constitute an equal and opposite couple. Find the reaction component Bx and the total force A on the supporting pin connection. Ans. Bx = 5710 lb., A = 7590 lb.

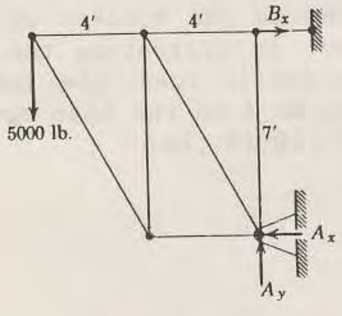


Fig. 4

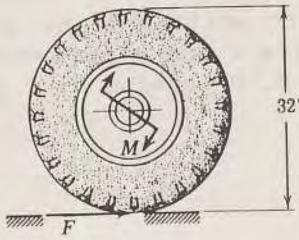


Fig. 5

5. The friction force F and driving couple M which act on the wheel of an accelerating truck may be replaced by a single equivalent force which acts through a point which is 0.4 in above the center for the wheel. If F equals 500 lb., determine the driving couple M supplied to the wheel through its drive shaft.

6. The control lever is subjected to a clockwise couple of 50 lb.ft. exerted by its shaft at A and is to be designed to operate with a 30 lb. pull as shown. If the resultant of the couple and the force passes through A, determine the proper dimension of the lever.

Ans. = 12.8 in.

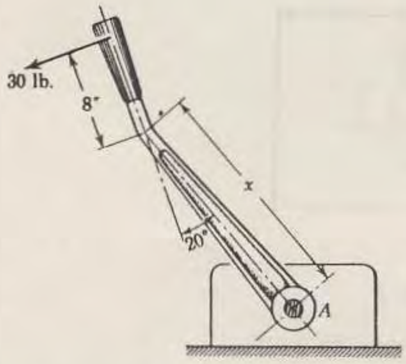


Fig. 6

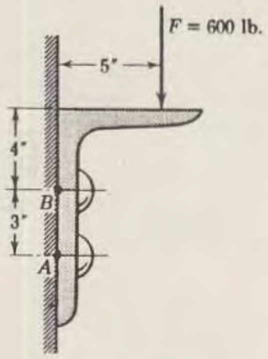


Fig. 7

7. The riveted angle bracket is held in equilibrium by the action of forces which are the equivalent of a force equal and opposite to and collinear with F. Represent these equivalent forces by a vertical force acting on the bracket at A and two horizontal forces P acting on the bracket, one at A and the other at B.

8. The 1000 lb. load is applied to the bracket attached to the free end of fixed cantilever I-beam. In evaluating the strength of the beam to withstand the applied eccentric load, the load may be replaced by its x and y components acting at A on the beam center line and a couple M. Determine M. Ans. M = 210 lb.,in.

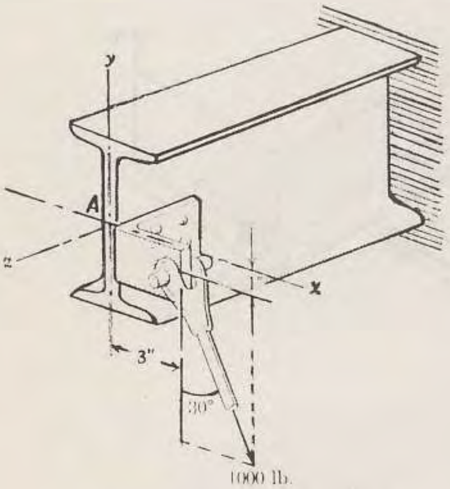


Fig. 8

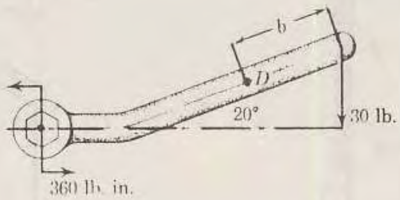


Fig. 9

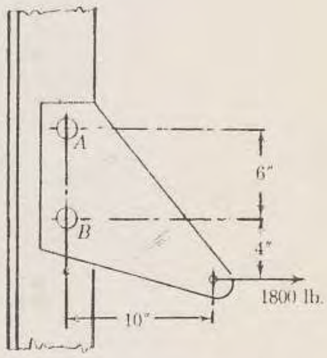


Fig. 10

9. Replace the couple and force shown by a single force F applied at a point D. Locate D by determining the distance b.

10. The bracket is fastened to the girder by means of the two rivets A and B. For equilibrium of the bracket the resultant of the forces exerted by the rivets on the bracket must be equal and opposite to and collinear with the 1800 lb. applied force. Find the force supported by each rivet by replacing the applied load by a force along a horizontal center line midway between the rivets and a couple and then by distributing this resulting system appropriately to the rivets. <u>Ans.</u> A = 1200 lb., B = 3000 lb.

11. Combine the force and the couple shown into a single force F, and find the perpendicular distance <u>d</u> from A to F.

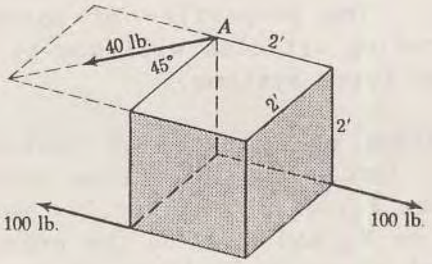


Fig.11

12. Determine the x, y, and z components of the single couple vector M which is equivalent to the two couples shown.

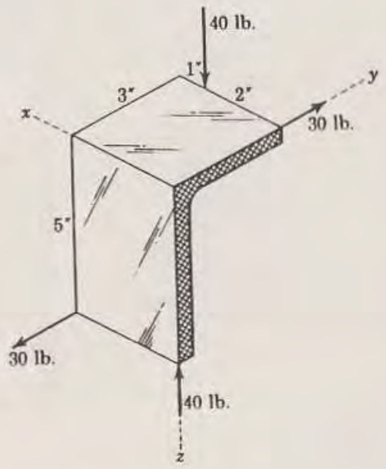


Fig. 12

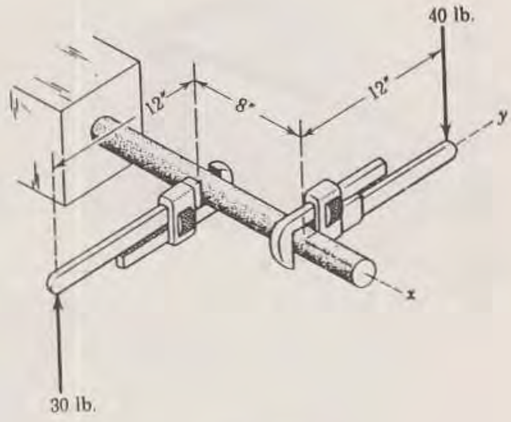


Fig. 13

13. Using the principle of transformation of a couple replace the two forces shown acting on the pipe wrenches by a single force F acting at a point P in the x-y plane. Find the coordinates of P.

Ans. F = 10 lb., x = 24 in., y = 84 in.

---

12. **Resultant of Coplanar Force Systems:** The resultant of a system of forces is the simplest force system which can replace the original forces without altering their external effect on a rigid body. The equilibrium of a body is the condition wherein the resultant of all forces is zero, and the accelerations of a body is described by equating the force resultant to the product of mass and acceleration. Thus the determination of resultants in basic to both statics and dynamics. The properties of force, moment, and couple discussed in the preceding articles will now be used to determine the resultants of coplanar force systems.

The resultant of a system of coplanar forces may be obtained by adding the forces two at a time and then combining their sums. The three forces F_1, F_2, F_3 of Fig.19a may be combined by first adding any two, such as F_2 and F_3. By the principle of transmissibility they may be moved along their lines of action to their point of concurrency A, and their sum R_1 formed by the parallelogram law. The force R_1 may then be combined with F_1 by the parallelogram law at their point of concurrency B to obtain the resultant R of the three given forces. The order of combination of the forces is immaterial as may be verified by combining them in a different sequence. The force R may be applied at any point on its established line of action.

The magnitude and direction of R may be obtained with addition by the triangle law as shown in Fig.19b. Here the forces are treated as free vectors and added head-to-tail. The resultant of F_1 and F_2 is a vector directed from 0 to 2 and when combined with F_3 gives R correct in magnitude and direction. The polygon 0-1-2-3 is known as a "FORCE POLYGON". Algebraically these results may be obtained by forming the rectangular component of the forces in any two

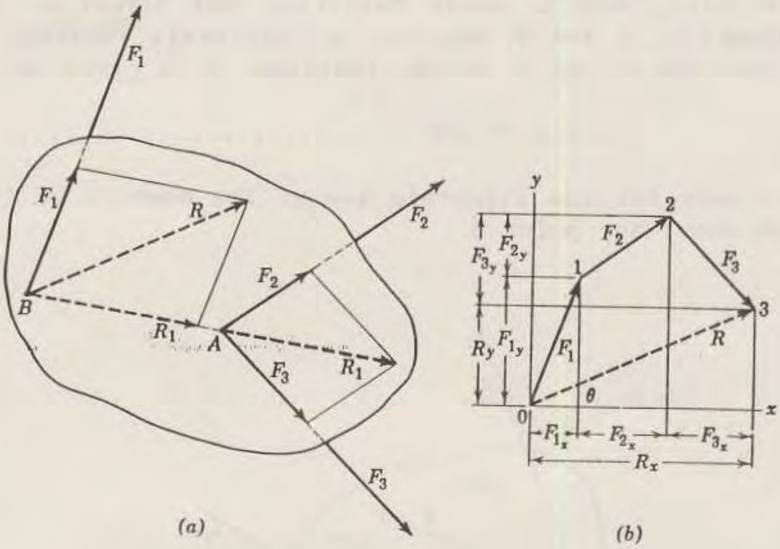


Fig.19

convenient perpendicular directions. In Fig.19b the x and y components of R are seen to be the algebraic sums of the respective components of the three forces. Thus, in general, the rectangular components of the resultant R of a coplanar system of forces may be expressed as

$$Rx = \Sigma Fx, \qquad Ry = \Sigma Fy,$$

$$\text{where} \qquad R = \sqrt{(\Sigma Fx)^2 + (\Sigma Fy)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

The angle made by R with the x-axis is clearly

$$\theta = \tan^{-1}\frac{\Sigma Fy}{\Sigma Fx} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

The location of the line of action of R may be computed with the aid of Varignon's theorem. Although this theorem was proved for two concurrent components of a given force, it holds for any system of forces. The moment of R, Fig.19a, about some point must equal the sum of the moments of its two components F_1 and R_1 about the same point. The moment of R_1, however, must equal the sum of the moments of its components F_2 and F_3 about the same point. It follows that the moment of R about any point equals the sum of the moments of F_1, F_2, and F_3 about this same point. Application of this "PRINCIPLE OF MOMENTS" about the point O, Fig.20, gives the equation

$$R.d = F_1d_1 - F_2d_2 + F_3d_3$$

for this particular configuration of forces where the clockwise direction is arbitrarily taken as positive. The distance d is computed from this relation, and R, whose magnitude and direction have been determined from Eqs. 7 and 8, may now be completely located. In general, then, the moment arm d. of the resultant R is given by

$$R.d = \Sigma Mo \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

where ΣMo stands for the algebraic sum of the moments of the forces of the system about any point O.

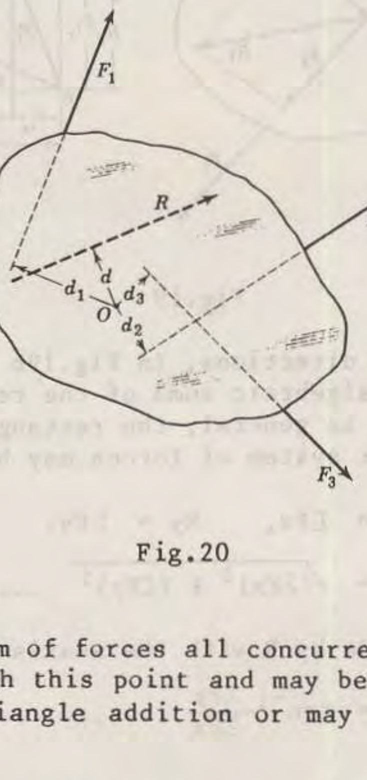


Fig.20

For a system of forces all concurrent at a given point the resultant passes through this point and may be determined graphically by parallelogram or triangle addition or may be computed from Eqs.7 and 8.

For a system of parallel forces the magnitude of the resultant is the algebraic sum of the several forces, and the position of its line of action may be obtained from the principle of moments expressed by Eqs.9.

Consider now a force system such as that shown in Fig. 21, where the polygon of forces closes and consequently there is no resultant force R. Direct combination by the parallelogram law shows that for the case illustrated the resultant is a couple of magnitude $F_3 . d$. The value of the couple is equal to the moment sum about any point. Thus it is seen that the resultant of a coplanar system of forces may be either a force or a couple.

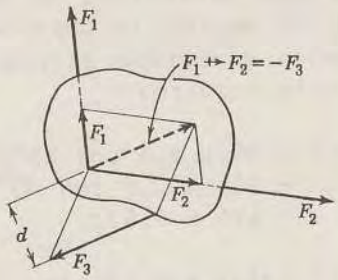
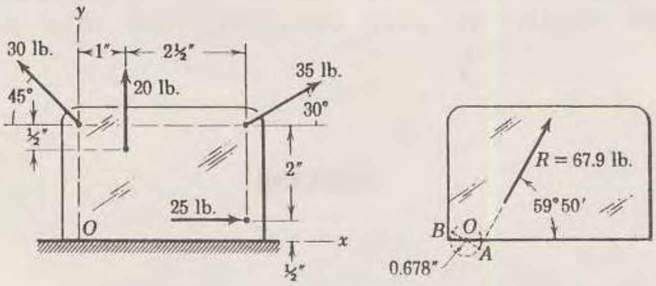


Fig.21

SAMPLE PROBLEM

Determine the resultant of the four forces acting on the plate shown.



Solution: Point O is selected arbitrarily as the origin of coordinates for an algebraic solution. The components Rx and Ry, the resultant R, and the angle θx become

$[R_x = \Sigma F_x]$: $R_x = -30 \cos 45° + 25 + 35 \cos 30° =$ 34.1 lb.,

$[R_y = \Sigma F_y]$: $R_y = 30 \sin 45° + 20 + 35 \sin 30° =$ 58.7 lb.,

$[R = \sqrt{R_x^2 + R_y^2}$: $R = \sqrt{(34.1)^2 + (58.7)^2} =$ 67.9 lb. Ans.

$[\theta_x = \tan^{-1} \frac{R_y}{R_x}]$: $\theta_x = \tan^{-1} \frac{58.7}{34.1} = 59° \; 50'$ Ans.

The resultant is now determined in magnitude and direction. The position of its line of action is obtained from the principle of moments. With the clockwise direction as positive and point O as the moment center this principle requires:-

$$[R.d = \Sigma M_o] : 67.9\, d = (-30 \times 2.5 \cos 45°) - (20 \times 1) + (35 \times 2.5 \cos 30°) - (35 \times 3.5 \sin 30°) + (25 \times 0.5)$$

$$d = -\frac{46.0}{67.9} = -0.678 \text{ in.}$$ Ans.

The negative sign indicates that the moment of the sum is counterclockwise since the clockwise direction was taken as positive. Hence the resultant R may be applied at any point on a line making an angle of 59° 50' with the x-axis and tangent to a circle of radius 0.678 in. about O as shown in the figure to the right of the drawing. The counterclockwise moment requires that the line of action of R be tangent at point A and not at point B as would have been the case if the moment had been positive in a clockwise sense.

Direct combination of the forces by the parallelogram law will yield the same result as just obtained, and this may be verified graphically.

PROBLEMS

1. The fixed eyebolt supports the tensions of three cables, indicated by the force vectors shown. If these three cables are removed and their combined effect on the bolt is achieved by a single cable, determine its tension T and direction θ_x. Ans. T = 1850 lb., $\theta_x = 22° \; 58'$.

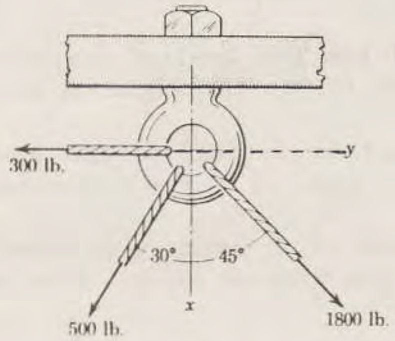


Fig. 1

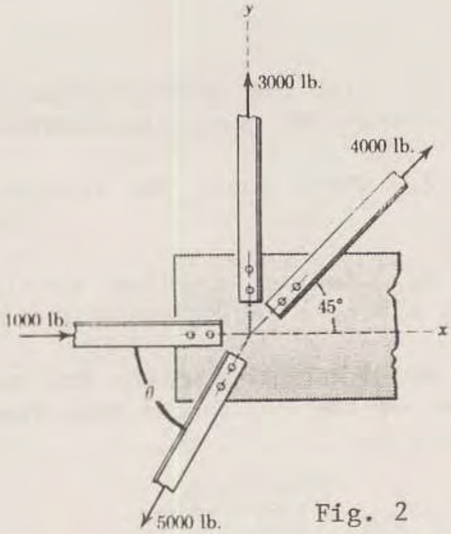


Fig. 2

2. Determine the resultant R of the four forces transmitted to the gusset plate if θ = 50° <u>Ans</u>. R = 2090 lb., θx = 72° 56'.

3. Determine the angle θ for the figure in problem 2 which will yield zero resultant force in the x-direction. <u>Ans.</u> θ = 40° 2'.

4. Determine the resultant R of the four loads acting on the Pratt bridge truss.

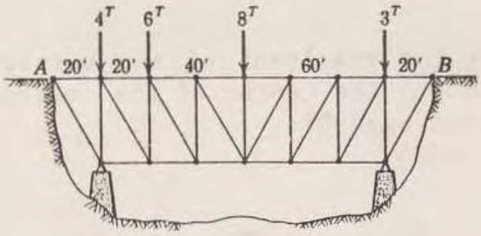


Fig. 4

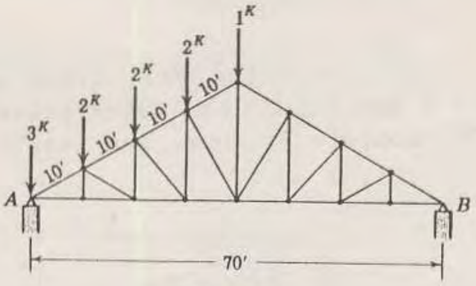


Fig. 5

5. Determine the resultant R of the roof loads shown acting on the Howe truss. <u>Ans</u>. R = 10 kips, 14 ft. to the right of A.

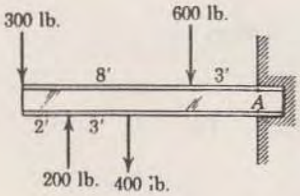


Fig. 6

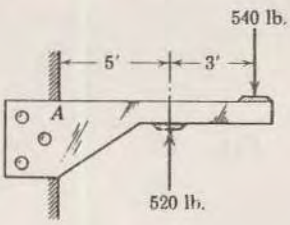


Fig. 7

6. Find the force R which could replace the four forces shown and not change the resulting reactions at the support A.

7. Where would the resultant of the two applied forces act?
Ans. 86 ft. to the right of A.

8. Replace the two parallel forces acting on the control lever by a single equivalent force R. Ans. R = 30 lb. through O.

9. Find graphically the magnitude of F which will make the resultant of the two forces pass through the bearing at O. Also solve algebraically.

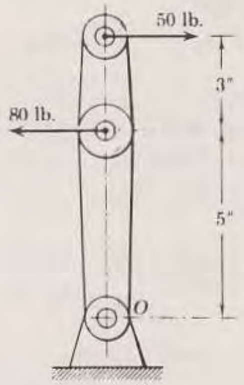


Fig. 8

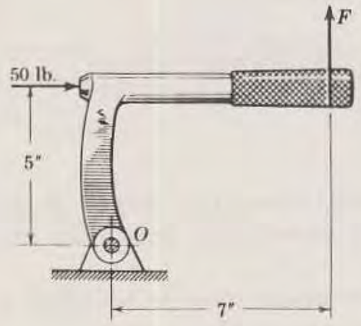


Fig. 9

10. For the bell crank shown the resultant of the vertical force F and the 30 lb. force passes through the center of the bearing at O. Compute F. Does the result depend upon θ?

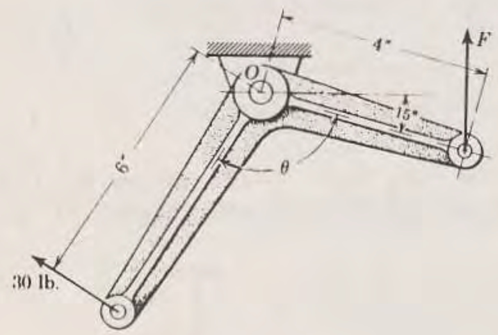


Fig. 10

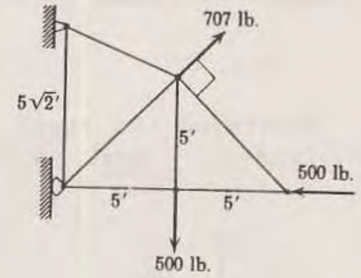


Fig. 11

11. Determine the resultant of the three forces acting on the truss. Ans. M = 2500 lb. ft., clockwise couple.

12. A group of three piles supports the load under a foundation wall. If each pile supports 100 kips in the direction of the pile, determine the distance x to the point P through which the resultant of the three loads passes, and find the angle θ made by the resultant with the vertical.

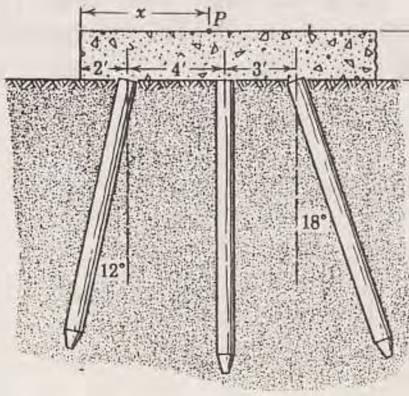


Fig. 12

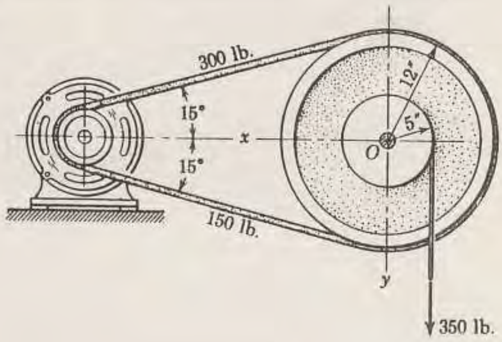


Fig. 13

13. Determine algebraically the resultant R of the two belt tensions and the 350 lb. load acting on the pulley and attached drum of the hoist. Find the distance d from R to O and specify the angle θ made by R with the x-axis. Ans. R = 583 lb., d = 0.0857 in., θx = 41° 50'.

14. The gear and attached V-belt pulley are being accelerated under the action of the 50 lb. gear-tooth force and the two belt tension of 36 lb. and 10 lb. Represent the resultant of the three forces by a force R passing through O and a couple M. What is the direction of the angular acceleration?

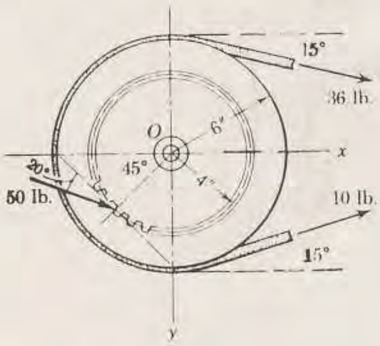


Fig. 14

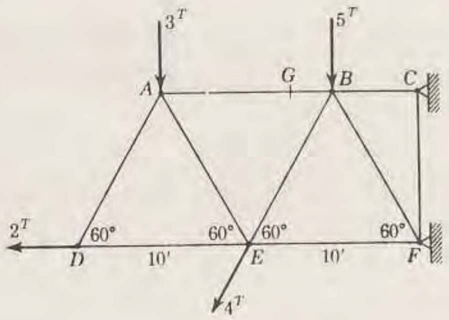


Fig. 15

15. Calculate the resultant R of the four forces acting on the cantilever truss and locate the point G on line AB through which it must pass. Ans. R = 12.14 tons through G 1.11 ft. to the left of B.

16. Determine the desirable width b. of the concrete abutment footing which will place the resultant of the four loads through point A which is 40% of the total base dimension to the left of point O. The 9 and 9.6 kip loads and the force W represent the weights of their respective portions of the abutment and pass through the centers of each rectangle. The weight W equals 1.20 b where b is expressed in feet. Ans. b = 3.4 ft.

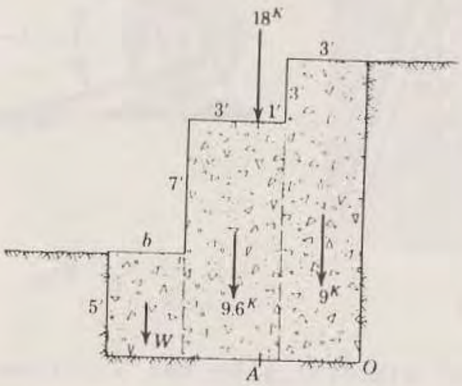


Fig.16

17. Represent the resultant of the applied loads and couple shown acting on the cantilever beam and bracket by a force R acting at A and a couple M.

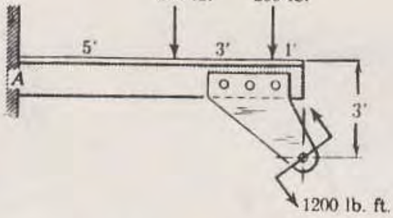


Fig.17

18. The gear reducer shown is subjected to the two couples, its 50 lb. weight, and a vertical force at each of the mounting A and B. If the resultant of this system of two couples and three forces is zero, determine forces A and B. Ans. A = 69.2 lb. down,
B = 119.2 lb. up.

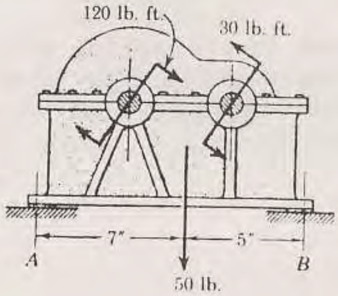


Fig. 18

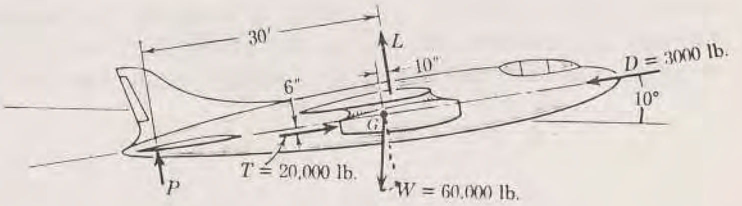


Fig. 19

19. The jet-propelled airplane is accelerating in the direction of its line of flight, and therefore the resultant of the external forces shown acting on it passes through its center of gravity G and is in the direction of flight. The total weight of the airplane is 60,000 lb., the air resistance or drag is 3000 lb., and each of its two jet engines develops a thrust of 10,000 lb. Determine the lift force L and the stabilizer force P for the conditions given. Ans. L = 57,100 lb., P = 1990 lb.

20. Determine the values of F, P, and θ so that the resultant of the three forces and couple shown is zero. Ans. F = 12.84 lb.,
P = 89.1 lb.,
θ = 4° 8'.

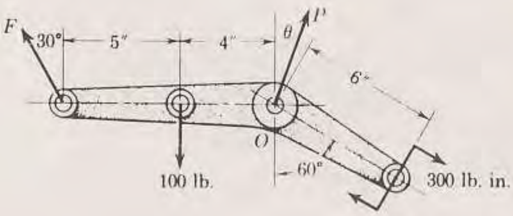


Fig. 20

13. **Resultants of Three-Dimensional Force Systems.** There are many engineering problems which require analysis in three dimensions. Such analysis calls for representation of the system by a pictorial drawing or by means of two or more orthographic projections. Facility to visualize and represent the third dimension is an absolute necessity for this type of analysis.

Consider first a system of concurrent forces acting on a body. An example of such a system is shown in the Fig.22a, where three forces not in the same plane act at O. The resultant R of this system may be obtained by adding the forces head-to-tail in space as

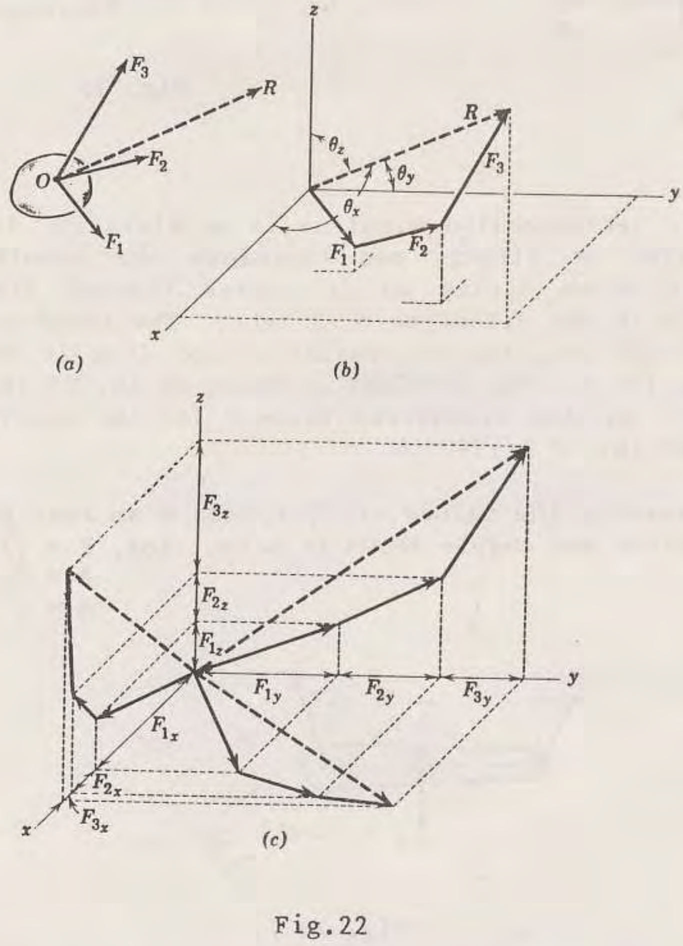


Fig.22

shown in Fig.22b. The resultant is the vector which closes the space polygon, and its three rectangular components are seen to be the algebraic sum of the corresponding components of the given forces.

Thus, in general; $Rx = \Sigma Fx$, $Ry = \Sigma Fy$, $Rz = \Sigma Fz$,

$$\text{and} \quad R = \sqrt{(\Sigma Fx)^2 + (\Sigma Fy)^2 + (\Sigma Fz)^2} \quad \text{.............. (10)}$$

If the angles made by R with the x, y, and z axes are θx, θy, and θz, respectively, then the direction of R is specified by its direction cosines

$$\text{Cos } \theta x = \frac{\Sigma Fx}{R}, \text{ Cos } \theta y = \frac{\Sigma Fy}{R}, \text{ Cos } \theta z = \frac{\Sigma Fz}{R} \quad \text{.............. (11)}$$

The space polygon in Fig.22b may be projected onto the three coordinate planes, giving three related plane polygons as shown in Fig.22c. Any two of these force polygons will involve all three of the first of equation (10), and hence two projections are sufficient for the full description of the force relations.

Now consider a general system of nonconcurrent forces in space acting on a body. An example of such a system is represented in Fig.23a, where three such forces are shown. By the principles of the Resolution of a Force into a Force and a Couple, the external effect on the body will be unaltered by applying F_1 at some other point, such as O, and adding the necessary couple M_1. The magnitude of this couple is $F_1.d$, where d is the perpendicular distance from O to the original line of action of F_1. This transfer for force F is shown in

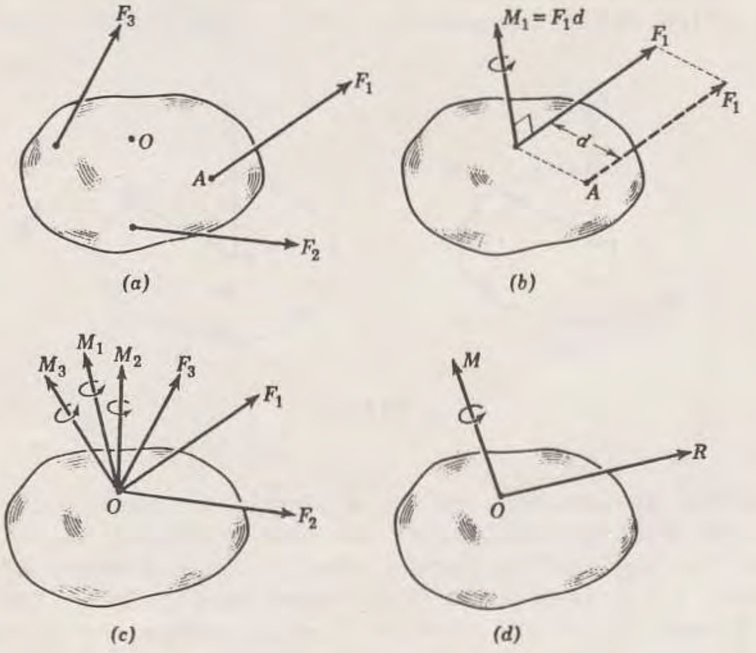


Fig.23

Fig.23b. The plane of the couple is the plane defined by the original line of action of F_1 and point O. The vector M_1 representing the couple is normal to this plane, and the sense is determined by the right-hand rule. The couple M_1 is a free vector but is shown at point O for convenience. The remaining two forces are transferred to point O with the addition of their two corresponding couples as shown in Fig.23c. The original force system has now been reduced to a system of concurrent forces at the arbitrary point O and a corresponding number of couples. The forces may be combined to get their resultant R acting at point O, and couples are combined vectorially to get a resultant couple M as shown in Fig.23d. For a given force system the couple M will vary in magnitude and direction, depending on the choice of the reference point O, although the resultant force R remains fixed in magnitude and direction regardless of the reference point.

It has now been shown that "the resultant of any general force system can be expressed in terms of a single resultant force and a single resultant couple" Physically the effects may be described as a push (or pull) R and a twist M. This result has a very important bearing on both the statics problem and the dynamics problem. In statics, where the equilibrium condition obtains, both R and M are zero. In dynamics, the linear and rotational accelerations will be dependent on R and M, respectively.

There are several special cases of general force system resultants which occur frequently. When there is no resultant force,

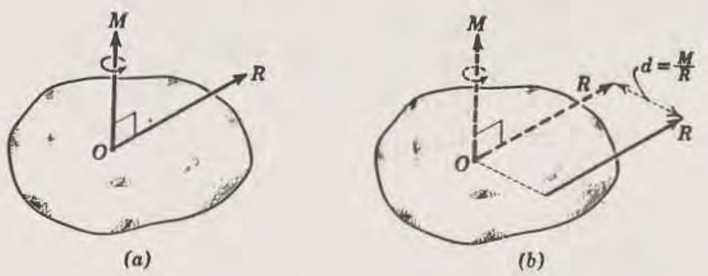


Fig.24

then the given system reduces to a couple or pure twist. When the couple vector M is perpendicular to the resultant force R, Fig.24a, the two may be combined to form a single force R whose line of action is a distance $d = \frac{M}{R}$ from point O as shown in fig.24b. The new line of action of R must be on the side of O which makes the direction of the moment of R about O the same as that of M. When the couple M is parallel to R, Fig.25, the combination is known as a Wrench or Screw. The action may be described as a push (or pull) and a twist about an

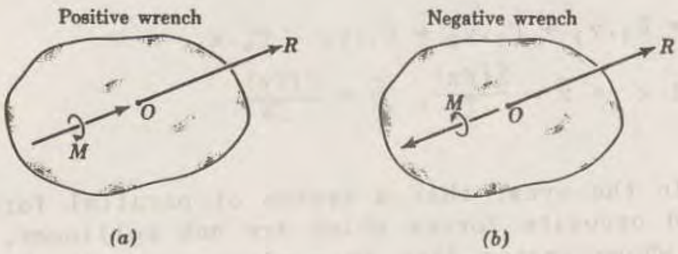


Fig.25

axis parallel to the push (or pull). When the force and moment vectors have the same sense, as in Fig.25a, the wrench is said to be positive. When the vectors have the opposite sense, the wrench is negative, as indicated in Fig.25b.

If the given forces are all parallel, the resultant R, if a force, is the algebraic sum of the several forces, and its position may be determined by the principle of moments. Thus for the 4 parallel forces shown acting on the plate of Fig.26 the resultant is:

$$R = F_1 + F_2 + F_3 - F_4 = \Sigma F.$$

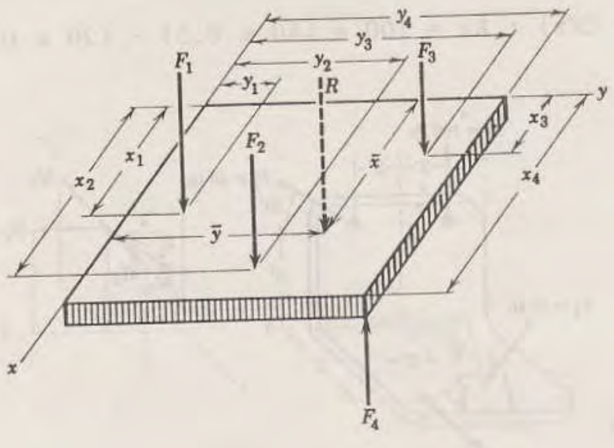


Fig.26

By the principle of moments the line of action of R is located by equating the moment of the sum to the sum of the moments about the y and x-axes. Thus:

$$R\bar{x} = F_1.x_1 + F_2.x_2 + F_3.x_3 - F_4.x_4,$$

$$R\bar{y} = F_1.y_1 + F_2.y_2 + F_3.y_3 - F_4.x_4,$$

or in general $x = \bar{x} = \frac{\Sigma(Fx)}{\Sigma F}, \quad \bar{y} = \frac{\Sigma(Fy)}{\Sigma F}$

In the event that a system of parallel forces reduces to two equal and opposite forces which are not collinear, the resultant is a couple whose vector lies in a plane perpendicular to the given forces.

SAMPLE PROBLEM

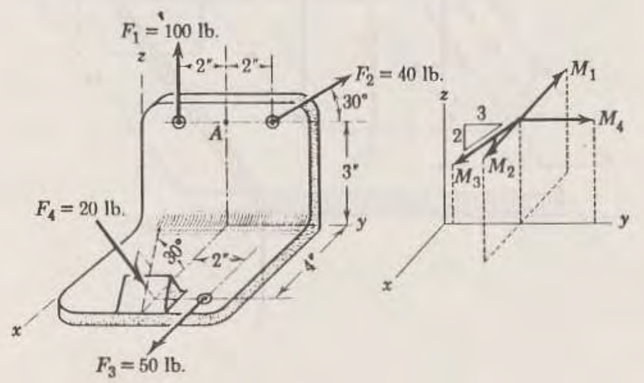
Replace the four forces acting on the bracket shown by a single equivalent force applied at point A and determine the corresponding couple which must also be applied.

Solution: The three rectangular components of the resultant force are:-

$[Rx = \Sigma Fx]$: $Rx = 50 - (20 \times 0.5) = 40$ lb.

$[Ry = \Sigma Fy]$: $Ry = (40 \times 0.866) = 34.6$ lb.

$[Rz = \Sigma Fz)$: $Rz = 100 + (40 \times 0.5) - (20 \times 0.866) = 102.7$ lb.



The resultant force is $R = \sqrt{Rx^2 + Ry^2 + Rz^2}$

$= \sqrt{(40)^2 + (34.6)^2 + (102.7)^2}$

$= 115.5$ lb. Ans.

The direction of R may be specified by its direction cosines:

$[\cos \theta x = \frac{Rx}{R}]$: $\cos \theta x = \frac{40}{115.5}$, $\theta x = 69° 45'$,

$[\cos \theta y = \frac{Ry}{R}]$: $\cos \theta y = \frac{34.6}{115.5}$, $\theta y = 72° 35'$,

$[\cos \theta z = \frac{Rz}{R}]$: $\cos \theta z = \frac{102.7}{115.5}$, $\theta z = 27° 15'$. Ans.

The couple due to the transfer of F to point A is:

$[M = F.d]$: $M_1 = 100 \times 2 = 200$ lb.in. (minus x-direction).

Likewise the couples due to the transfer of F_2, F_3, and F_4 are

$M_2 = 40 \times 2 \times 0.5 = 40$ lb.in. (plus x-direction)

$M_3 = 50 \times \sqrt{2^2 \times 3^2} = 180.3$ lb.in. (y-z plane)

$M_4 = (20 \times 4 \times 0.866) + (20 \times 3 \times 0.500) = 99.3$ lb.in. (plus y-direction)

These four couple vectors are shown in the right-hand part of the figure. The three rectangular components of the resultant couple are seen from the figure to be:-

$Mx = -200 + 40 = -160$ lb.in.,

$My = 99.3 - (180.3 \times \frac{3}{\sqrt{2^2 + 3^2}}) = -50.7$ lb.in.

$Mz = -180.3 \times \frac{2}{\sqrt{2^2 + 3^2}} = -100$ lb.in.

The resultant couple is:-

$[M = \sqrt{Mx^2 + My^2 + Mz^2}]$: $M = \sqrt{(160)^2 + (50.7)^2 + (100)^2}$

$= 195.3$ lb.in. Ans.

The direction of the couple may be specified by its direction cosines, which are

$[\cos \theta x = \frac{Mx}{M}]$: $\cos \theta x = \frac{-160}{195.3}$, $\theta x = 145° 0'$,

$[\cos \theta y = \frac{My}{M}]$: $\cos \theta y = \frac{-50.7}{195.3}$, $\theta y = 105° 05'$,

$[\cos \theta z = \frac{Mz}{M}]$: $\cos \theta z = \frac{-100}{195.3}$, $\theta z = 120° 50'$. Ans.

The corresponding acute angles made by M with the negative directions of the x, y, and z coordinate axes are 35° 0', 74° 55', and 59° 10', respectively.

It should be noted that the moment M_3 due to F_3 could have been expressed in terms of its two components, $M_{3}y$ = -150 lb.in., and $M_{3}z$ = -100 lb.in., since M is most easily computed from the rectangular components of the moment vectors.

PROBLEMS

1. The concrete slab supports the six vertical loads shown. Determine the resultant of these forces and the x and y coordinates of a point through which it acts.

Ans. R = 27 tons, x = 7 ft., y = 11.92 ft.

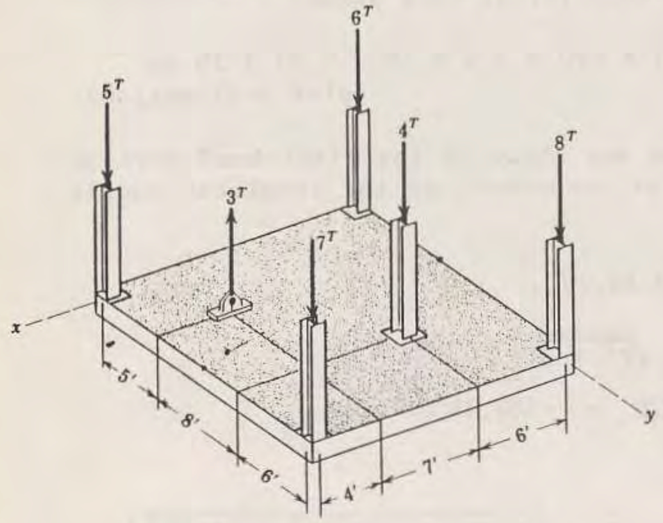


Fig. 1

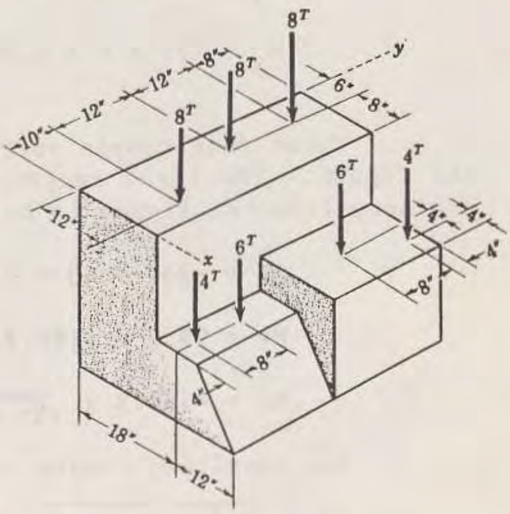


Fig. 2

2. The concrete abutment supports the seven loads shown. Determine the resultant R of these forces and specify the coordinates of a point on its line of action.

Ans. R = 44 tons, x = 13.8 in., y = 21.5 in.

3. Determine the resultant of the four forces and the couple which act on the shaft.

<u>Ans.</u> Resultant is a couple M = 589 lb.ft.

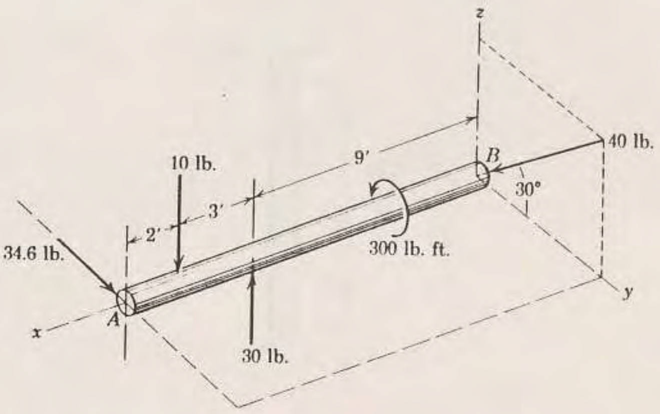


Fig. 3

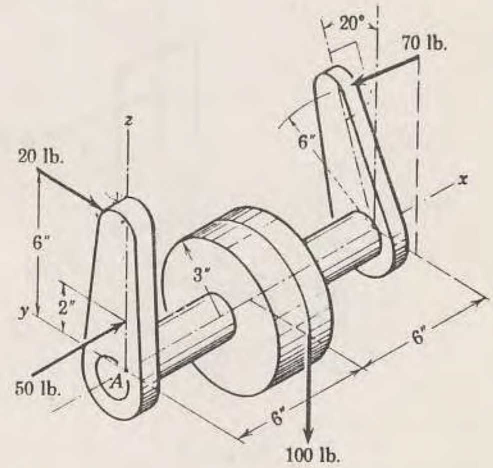


Fig. 4

4. Replace the four forces shown by an equivalent system consisting of a force R at A and a couple M.

5. Find the resultant of the two forces shown, using point A as the point through which the resultant force shall pass.

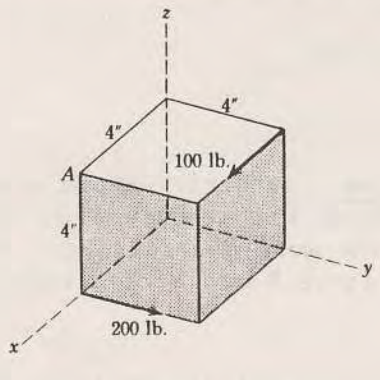


Fig. 5

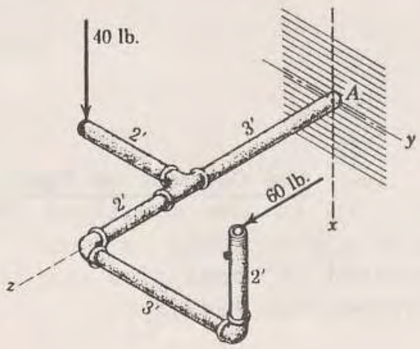


Fig. 6

6. The rigid piping shown is built into the wall and is subjected to the two forces indicated. Determine the resultant force R and moment M at point A of these two applied loads.

7. Replace the two forces shown by a single force R acting at A and a couple M. Ans. R = 128 lb., M = 610 lb.in.

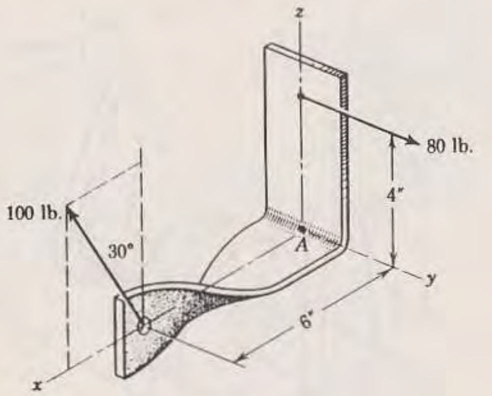


Fig. 7

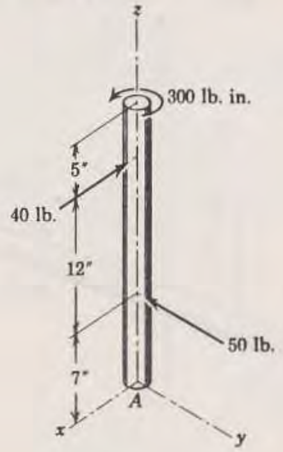


Fig. 8

8. Replace the couple and two applied forces shown acting on the vertical shaft by a resultant force R and couple M acting at point A.

EQUILIBRIUM

14. **The Concept of Equilibrium:** is derived from a balance of forces. It is the condition for which the resultant of all forces acting on a given body is zero. As the most general force system may be expressed in terms of a resultant force R and a resultant couple M. Equilibrium thus requires:

$$R = 0 \quad , \quad M = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

Equations 12 are the necessary conditions for equilibrium. Physically these vector equations mean that for a body in equilibrium there is as much force acting on it in one direction as in the opposite direction and that there is as much twist or moment applied to it about an axis in one sense as in the opposite sense. Thus equilibrium implies a balance of forces and a balance of moments. Graphically Eqs. 12 require that the space polygon of forces and the space polygon of corresponding couple vectors shall both close.

In order for a body to be in complete equilibrium it is necessary for both of Eqs. 12 to hold. These equations, however, represent two independent conditions, and either may hold without the other. If $R = 0$ and $M = 0$, the center of gravity of the body either is at rest or is moving with a constant velocity, and the body has an angular acceleration. Under these conditions it may be said that the body is in equilibrium only in so far as its linear motion is concerned. On the other hand if $R = 0$ and $M = 0$, the center of gravity of the body has a linear acceleration but there is no rotational acceleration. Such a body may be considered to be in rotational equilibrium. If the linear acceleration of a body is in the x-direction, then the body may be considered to be in equilibrium in the y-and z-directions since it has no acceleration in these directions. The term equilibrium, however, is most commonly used to describe a body which is completely at rest, as implied by the term statical equilibrium.

15. **Free-Body Diagrams:** In using equation (12) it is necessary to account accurately and completely for all forces which are applied on the body in question. Error will result if one or more forces are not accounted for or if forces which do not act on the body in question are used. In analyzing the action of forces on a given body it is absolutely necessary to isolate the body in question by removing all contacting and attached bodies and replacing them by vectors representing the forces which they exert on the body isolated. Such a representation is called a "FREE BODY DIAGRAM". The free-body diagram is the means by which complete and accurate account of all forces acting on the body in question may be taken. Unless such a diagram is correctly drawn the effects of one or more forces will be easily omitted and error will result. The free-body diagram is a basic step in the solution of problems in mechanics and is preliminary to the application of the mathematical principles which govern the state of equilibrium or motion.

The subject of equilibrium involves application of the very basic physical laws expressed by Eq. 12. In words these laws describe a condition wherein there is no resultant push (or pull) and no resultant twist on a body in equilibrium. The physical meaning of these laws of equilibrium is elementary, and the application of these laws for analytical solutions of equilibrium problems will also become elementary if accurate account of all forces is taken. The free-body diagram is the means by which this accurate account is made. An understanding of the free-body diagram method is truly the key to the understanding of mechanics. This comprehension can not be gained without ample practice, but a convincing initial impression of the importance of this method of attack is extremely valuable to the beginner. When the student encounters difficulty in the problem work, the correctness of the free-body diagram is the first step to check.

The actual procedure for drawing free-body diagrams will now be explained. There are two essential parts to this procedure:

(1) A clear decision must be made as to exactly what body (or group of bodies considered as a single body) is to be isolated and analyzed. An outline representing the external boundary of the body selected is then drawn.

(2) All forces, known and unknown, which are applied externally to the isolated body should be represented by vectors in their correct position. Known forces should be labeled with their magnitudes, and unknown forces should be labeled with appropriate symbols. In many instances the correct sense of an unknown force is not obvious at the outset. In this event the sense may be arbitrarily assumed. The correctness or error of the assumption will become apparent when the algebraic sign of the force is determined upon calculation. A plus sign indicates the force is in the direction assumed, and a minus sign indicates that the force is in the direction opposite to that assumed.

16. **Equilibrium in Two Dimensions:** When a body is in equilibrium under the action of forces all of which act in a single plane, say the x-y plane, Eqs. 12 become:

$$R = \sqrt{(\Sigma Fx)^2 + (\Sigma Fy)^2} = 0,$$

$$M = Mz \qquad\qquad = 0.$$

These equations may be rewritten as:

$$\Sigma Fx = 0 \; ; \; \Sigma Fy = 0 \; ; \; \Sigma Mo = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

where ΣMo represents the algebraic sum of the moments of all forces acting on the body about an axis parallel to the z-direction and passing through any point O on the body or off the body but in the x-y plane. More frequently this summation is referred to as the sum of the moments about point O. Equations 13 are the three most commonly used equations in the subject of statics. Physically they express the fact that for any body in equilibrium under the action of a coplanar system of forces there is as much force acting in any one direction as in the opposite direction and there is as much twist or moment about any point in one sense as in the opposite sense. Graphically Eqs. 13 require that the polygon of forces must close (zero resultant force) and that the string polygon must also close (zero resultant couple). Eqs. 13 are the necessary conditions for two-dimensional equilibrium, and, as may be concluded, they are independent conditions. Any one of the relations may hold without the other when the equilibrium is not complete.

There are two additional ways of expressing the necessary conditions for the equilibrium of forces in two dimensions. For the body shown in Fig.27a, if $\Sigma MA = 0$, then the resultant R, if it still exists, cannot be a couple but must be a force R passing through A.

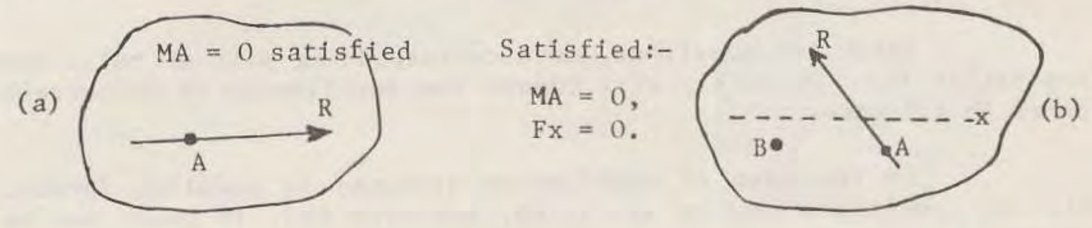


Fig. 27

If now the equation $\Sigma Fx = 0$ holds, where the x-direction is perfectly arbitrary, it follows from Fig.27b, that the resultant force R, if it still exists, must not only pass through A, but also must be perpendicular to the x-direction as shown. Now, if $\Sigma MB = 0$, where B is any point such that the line AB is not perpendicular to the x-direction, it is clear that R must be zero, and hence the body is in equilibrium. Therefore an alternate set of equilibrium equations is:

$$\Sigma Fx = 0 \ ; \ \Sigma MA = 0 \ ; \ \Sigma MB = 0 \quad \ldots\ldots\ldots\ldots \quad (14)$$

where the two points A and B are not on a line perpendicular to the x-direction.

A third formulation of the conditions of equilibrium may be made. Again, if $\Sigma MA = 0$ for any body such as shown in Fig.28a, the resultant, if it exists, must be a force R through A. In addition if $\Sigma MB = 0$, the resultant, if one still exists, must pass through B as shown in Fig.28b. Such a force cannot exist, however, if $\Sigma MC = 0$, where C is not collinear with A and B. Hence the equations of equilibrium may be written:

$$\Sigma MA = 0 \ ; \ \Sigma MB = 0 \ ; \ \Sigma MC = 0 \quad \ldots\ldots\ldots\ldots \quad (15)$$

where A, B, and C are any three points not on the same straight line.

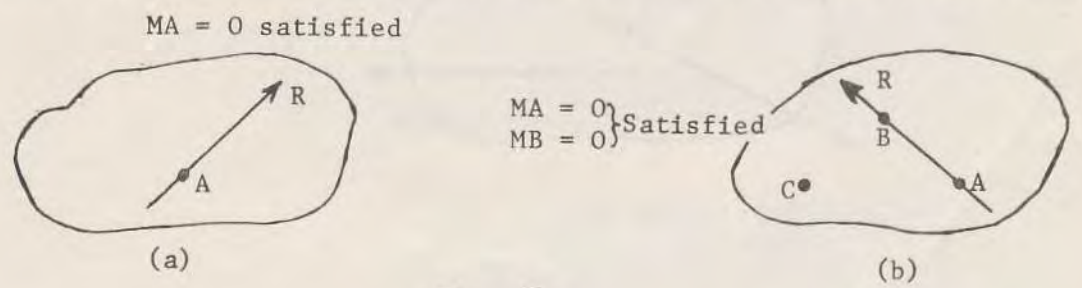


Fig. 28

When a body is in equilibrium under a system of concurrent forces, the moment sum about the point of concurrency is automatically satisfied, leaving from Eqs. 13 the relations:

$$\Sigma Fx = 0, \quad \Sigma Fy = 0$$

These two equations, or a corresponding pair of relations from either Eqs. 14 or 15, will insure the equilibrium of concurrent forces in all cases.

In the case of equilibrium produced by parallel forces, only two conditions need be specified, and from Eqs. 13 these may be taken to be:

$$\Sigma Fx = 0, \quad \Sigma Mo = 0$$

where the x-direction is the direction of the forces and O is any point in their plane. As an alternative two moment equations may be used, provided the line joining the moment centers is not parallel to the forces.

When a body is in equilibrium under the action of three forces only, these forces must be concurrent. If they were not concurrent, as shown in Fig.29, then one of the forces would exert a resultant moment about the point of concurrency of the other two which would violate the requirement of zero moment about every point. Any system of coplanar forces in equilibrium may always be reduced by direct combination of the forces to a system of three forces which must therefore be concurrent. This principle of concurrency is very useful for both algebraic and graphical solutions in determining the direction of an unknown force. The only exception to this principle exists for the case where three parallel forces are in equilibrium. In this event the point of concurrency may be said to be at infinity.

The equilibrium of collinear forces requires but one equation and is considered too trivial for further comment.

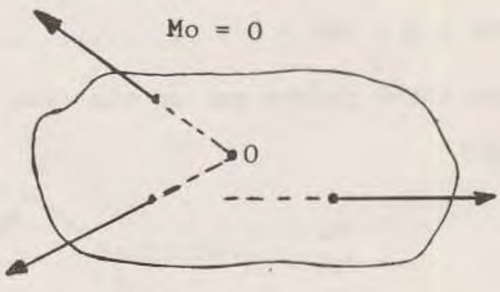
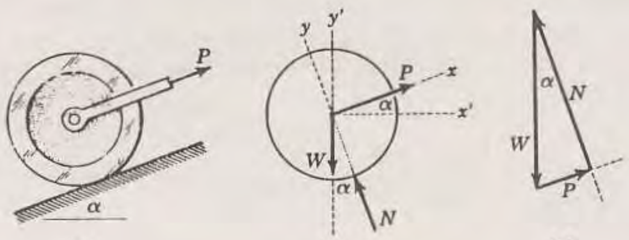


Fig.29

It is desirable that facility with both algebraic and graphical solutions be acquired in the problem work. The sample problems which follow illustrate both methods.

SAMPLE PROBLEMS

1. Determine the force P required to prevent the wheel of weight W from rolling down the incline. Also find the force N exerted by the plane on the wheel. Neglect the weight of the yoke.



__Algebraic Solution:__ The free-body diagram is first drawn as shown in part (b) of the figure. The weight W acts vertically down, and the force N is normal to the wheel surface. The diagram discloses the fact that the wheel is in equilibrium under the action of the three forces. Choosing the x and y directions along and normal to the plane, respectively, and applying the equations of equilibrium give:

$[\Sigma Fx = 0]$ $P - W \sin \alpha = 0, \quad P = W \sin \alpha$ __Ans.__

$[\Sigma Fy = 0]$ $N - W \cos \alpha = 0, \quad N = W \cos \alpha$ __Ans.__

If the axes were chosen as with x' and y', then the equations of equilibrium give:

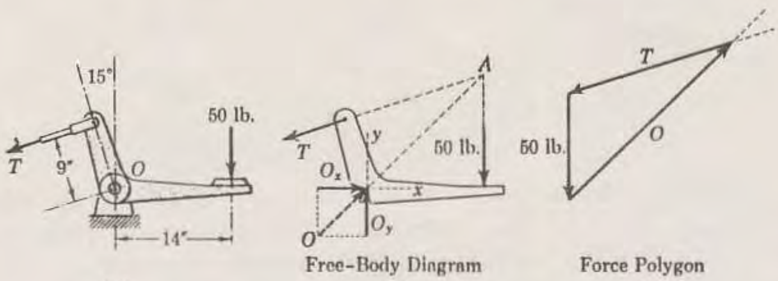
$[\Sigma Fx' = 0]$ $P \cos \alpha - N \sin \alpha = 0,$

$[\Sigma Fy' = 0]$ $P \sin \alpha + N \cos \alpha - W = 0.$

Eliminating N between the two equations and solving for P and then solving for N yield the answers just obtained, but the process was needlessly complicated by a less favourable choice of reference axes.

Graphical Solution: After the free-body diagram is completed the known force W is laid off to some convenient scale as shown in part (c) of the figure. Next lines with the known directions of P and N are constructed through the terminal points of W, and their intersection gives the solution and enables the correct magnitudes of P and N to be scaled from the drawing. It should be observed from this force polygon that the two answers may be obtained by inspection of the trigonometry of the triangle. Thus $P = W \sin \alpha$, $N = W \cos \alpha$, and also $P = N \tan \alpha$. In the case of concurrent forces a simple and approximate sketch of the force polygon will enable the required relations between the forces to be written immediately by inspection. It should also be noted that the force polygon is a graphical statement of $\Sigma Fx = 0$ and $\Sigma Fy = 0$.

2. A 50 lb. force is required to operate the foot pedal shown. Determine the tension T in the connecting link and the force exerted by the bearing O on the lever. The weight of the lever is negligible.



Algebraic Solution: The free-body diagram is first drawn as shown in part (b) of the figure. The force exerted by the bearing on the lever at O is shown in terms of its x and y components. The solution is best started with the moment equation about O, which eliminates Ox and Oy from the relation. Thus:-

$[\Sigma Mo = 0] \qquad 50 \times 14 - 9T = 0; \qquad T = 77.8 \text{ lb.}$ Ans.

The remaining two equations of equilibrium give:

$[\Sigma Fx = 0] \qquad Ox - 77.8 \cos 15° = 0, \qquad Ox = 75.1 \text{ lb.},$

$[\Sigma Fy = 0] \qquad Oy - 50 - 77.8 \sin 15° = 0, \qquad Oy = 70.1 \text{ lb.},$

$[O = \sqrt{Ox^2 + Oy^2}] \quad O = \sqrt{(75.1)^2 + (70.1)^2} \quad O = 102.7 \text{ lb.}$ Ans.

The direction of O can be specified by the angle determined by the ratio of Ox to Oy if desired.

Graphical Solution: The intersection of the 50 lb. force and the tension T defines the point A through which the reaction O must also pass, since three forces in equilibrium are concurrent. With the direction of O established, the force polygon is drawn in part (c) of the figure by constructing the directions of O and T through the ends of the 50 lb. vector. From the intersection of the two lines the magnitudes of T & O are scaled directly from the triangle, and the values are those obtained in the algebraic solution. In this problem the exact expressions for the unknown forces are more easily obtained from the algebraic solution than from the trigonometry of the force polygon.

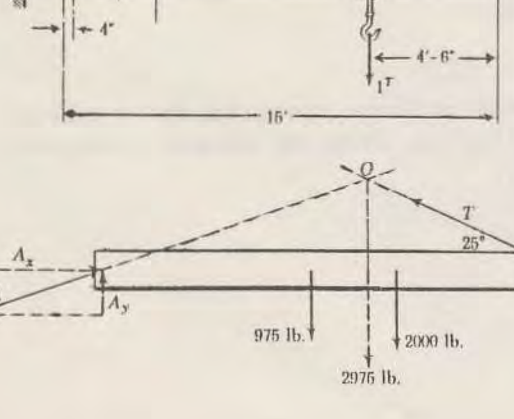
3. Determine the tension T in the supporting cable and the force on the pin at A for the jib crane shown. The beam AB is a standard 18 in. I-beam weighing 65 lb./ft. of length.

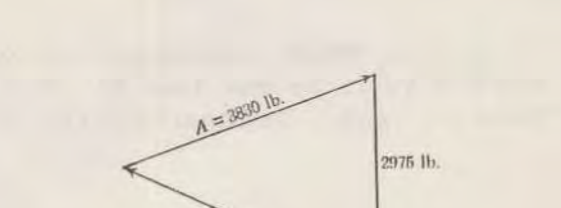
Algebraic Solution: The free-body diagram of the beam is shown in part (a) of the figure with the pin reaction at A separated in terms of its two rectangular components. The weight of the beam is 65 x 15 = 975 lb. and acts through its center. In applying the moment equation about A it is simpler to consider the moments of the x and y components of T than it is to compute the perpendicular distance from T to A. Thus:

$$[\ M_A = 0] : (T \cos 25°)\frac{9}{12} + (T \sin 25°)\ (15 - \frac{4}{12}) - 2000\ (15 - 4.5 - \frac{4}{12})$$

$$- 975\ (7.5 - \frac{4}{12}) \quad = \quad 0,$$

from which : $T \quad = \quad 3980$ lb. <u>Ans.</u>





(a) Free-Body Diagram

(b) Graphical solution

Equating the sum of forces in the x and y directions to zero gives:

$[\Sigma Fx = 0] : Ax - 3980 \cos 25° = 0, \quad Ax = 3610$ lb.,

$[\Sigma Fy = 0] : Ay + 3980 \sin 25° - 975 - 2000 = 0, Ay = 1293$ lb.,

$[A = \sqrt{Ax^2 + Ay^2}] : A = \sqrt{(3610)^2 + (1293)^2}, A = 3830$ lb. Ans.

Graphical Solution: The principle that three forces in equilibrium must be concurrent is utilized for a graphical solution by combining the two known vertical forces of 975 lb. and 2000 lb. into a single 2975 lb. force located as shown on the modified free-body diagram of the beam in part (b) of the figure. The position of this resultant load may be determined graphically or algebraically.

The intersection of the 2975 lb. force with the line of action of the unknown tension T defines the point of concurrency O through which the pin reaction A must pass. The unknown magnitudes of T and A may now be found by constructing the closed equilibrium polygon of forces. After laying off the known vertical load to a convenient scale, as shown in the lower part of the figure, a line representing the given direction of the tension T is drawn through the tip of the 2975 lb. vector. Likewise a line representing the direction of the pin reaction A, determined from the concurrency established with the free-body diagram, is drawn through the tail of the 2975 lb. vector. The intersection of the lines representing vectors T and A establishes the magnitudes of both T and A which are necessary to make the vector sum of the forces equal to zero. These magnitudes may be scaled directly from the diagram. The x and y components of A may be constructed on the force polygon if desired.

It should be noted that the polygon is begun with the known forces first and that the order of addition of the remaining two vectors is immaterial.

PROBLEMS

1. Which combination of rope and pulleys requires the smaller force P to raise the load W? Each pulley is free to rotate independently. Ans. Combinations (b); $P = \frac{W}{5}$.

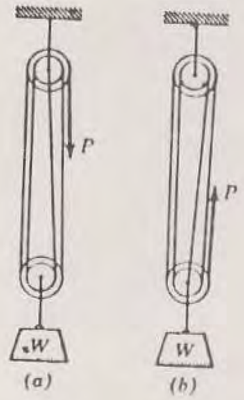


Fig. 1

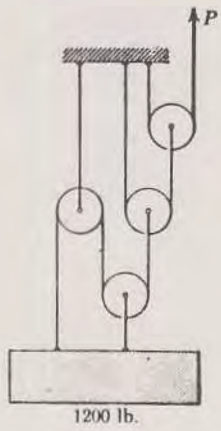


Fig. 2

2. What force P is required to raise the 1200 lb. platform?
Ans. P = 100 lb.

3. The 6½ yard concrete truck is weighed on a highway scale, and it is noted that the front axle supports 14,000 lb. and each of the rear axles supports 16,000 lb. Determine the distance x from the front axle to the center of gravity of the loaded truck.
Ans. x = 107 in.

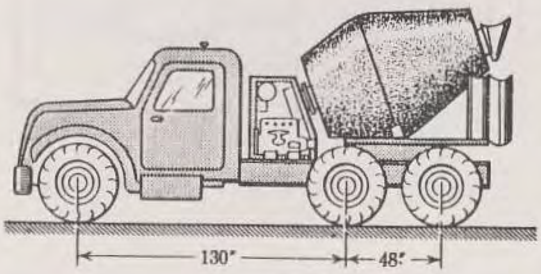


Fig.3

4. Determine the horizontal force P applied at point A on the cable necesary to position the 1000 lb. load directly over the opening. Ans. P = 258 lb.

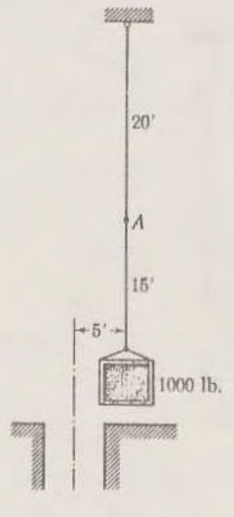


Fig.4

5. One type of pipe cutter, shown in the figure, consists of a clamp with three sharp-edged wheels which cut through the pipe as the screw is tightened and the device is rotated around the pipe.

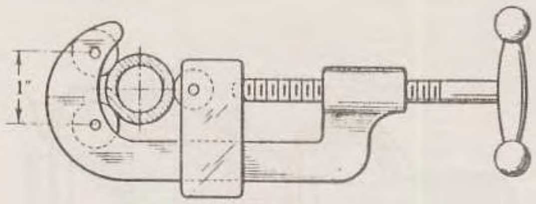


Fig.5

When cutting a $\frac{3}{4}$ in. pipe (outside diameter 1.05 in.) with a compression of 1000 lb. in the screw, determine the force R acting on each pin of the two $\frac{3}{4}$ in. diameter cutting wheels. Ans. R = 602 lb.

6. A 300 ton refinery tower with center of gravity at section G is to be lifed from the freight cars. Two slings at A and one at B are to be used. If it necessary to limit the tension T_2 in each cable at B to 25 tons, determine the minimum distance x.

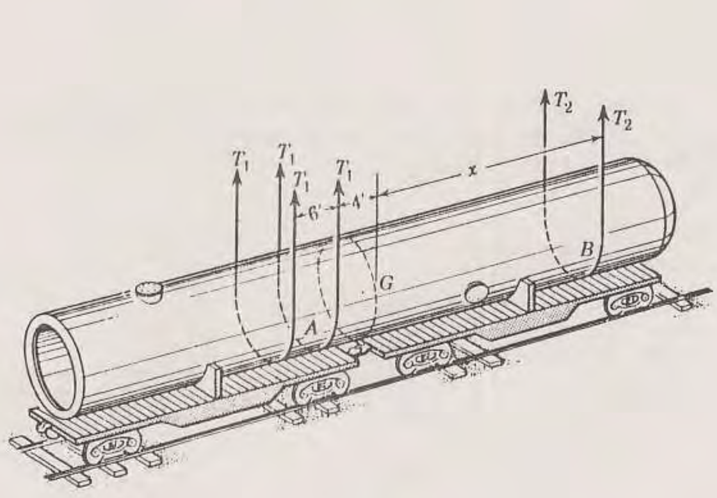


Fig.6

7. In designing the bracket to support the 1500 lb. load the greatest shear force which either bolt will safely support is 5000 lb. Determine the minimum distance x.

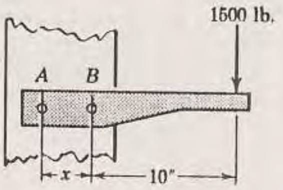


Fig. 7

Fig. 8

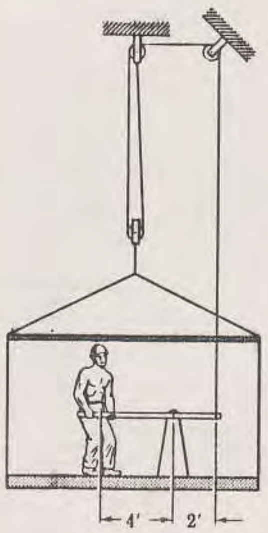


Fig. 9

8. A painter weighing 150 lb. hoists himself on a bosun's chair as shown. Determine the force P which he must apply to the rope in order to support himself. Ans. P = 50 lb.

9. A man weighing 160 lb. hoists himself and the 200 lb. platform by pulling up on the lever shown. Determine the reaction R between his feet and the platform.

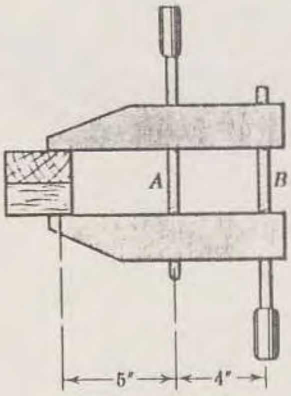


Fig. 10

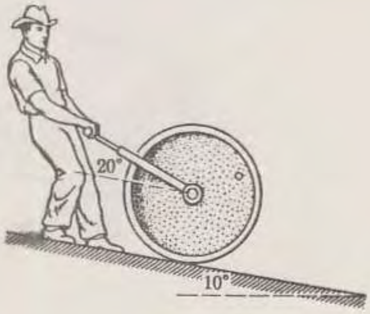


Fig. 11

10. If the wood clamp holds the two pieces together with a force of 100 lb., what are the vertical forces FA and FB in the two screws? Ans. FB = 125 lb., FA = 225 lb.

11. What force P must be applied to the 200 lb. lawn roller to prevent it from gathering speed as it is rolled down the 10 degree incline?

12. Determine the pull P required to lift the load W for the differential chain hoist shown. The two upper pulleys are fastened together, and the chain can not slip on the pulleys.

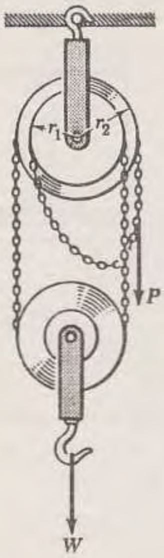


Fig. 12

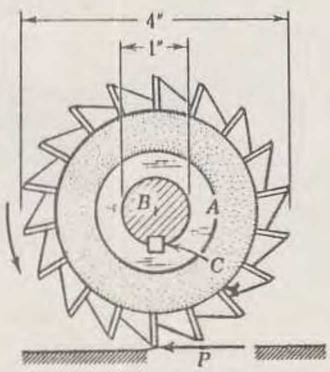


Fig. 13

13. The milling cutter A is prevented from turning on its arbor B by means of a square key which bears horizontally against the cutter at point C in the position shown. If the motor drive supplies a torque of 100 lb.ft. to the arbor, determine the force F transmitted by the key to the cutter, the reaction R between the arbor and the smooth inner periphery of the cutter, and the tooth load P.

<u>Ans</u>. F = 2400 lb., R = 1800 lb., P = 600 lb.

14. Determine the least torque M which must be applied to the axle of the 20 lb. wheel to cause it to roll over the obstruction if the wheel does not slip. Calculate the corresponding friction force F (tangent to the wheel surface) exerted on the wheel at A by the obstruction.

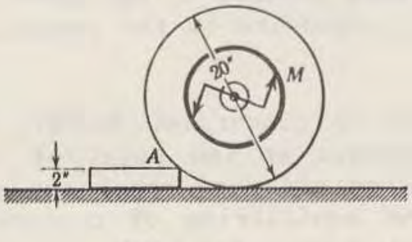


Fig. 14

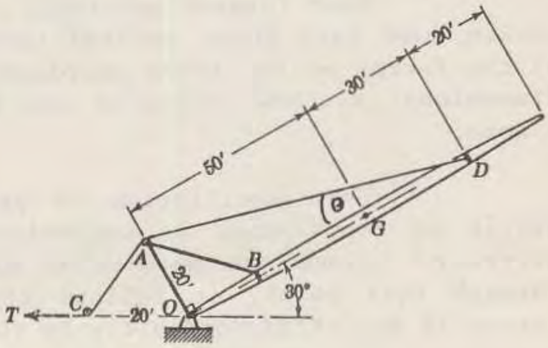


Fig. 15

15. The 100 ft. mast with center of gravity at G. weights 1800 lb. and is erected as shown. The cable passes over the pulleys at C and A and is secured to the mast at D. Determine algebraically the tension T required to support the mast in the 30 deg. position. Neglect the diameters of the two pulleys and also the weight of the struts OA and AB. Discuss any effect on T caused by securing the cable to point A in place of using a pulley at this point.

<u>Ans</u>. T = 4500 lb.

---

17. <u>**Equilibrium in Three Dimensions:**</u> When applied to the general three-dimensional problem, the vector equations of equilibrium Eqs.12, may be written in scalar form as:

$$\Sigma Fx = 0, \quad \Sigma Mx = 0,$$
$$\Sigma Fy = 0, \quad \Sigma My = 0, \quad \ldots\ldots\ldots\ldots \quad (16)$$
$$\Sigma Fz = 0, \quad \Sigma Mz = 0,$$

The first three of these six equations state that for a body in equilibrium there is no resultant force acting on it in any of the three directions. The second three equations express the requirement that there is no resultant twist or moment about any of the coordinate axes or about axes parallel to the coordinate axes. These six equations are the necessary conditions for complete equilibrium. Equations 16 are entirely independent, and any of them may hold without the others, in which case the body would be in partial equilibrium only. If, for instance, $\Sigma Mx = 0$, but the remaining five equations are satisfied, this combination would describe a body, such as a wheel, fixed at its center of gravity but accelerating in its rotation about the x-axis.

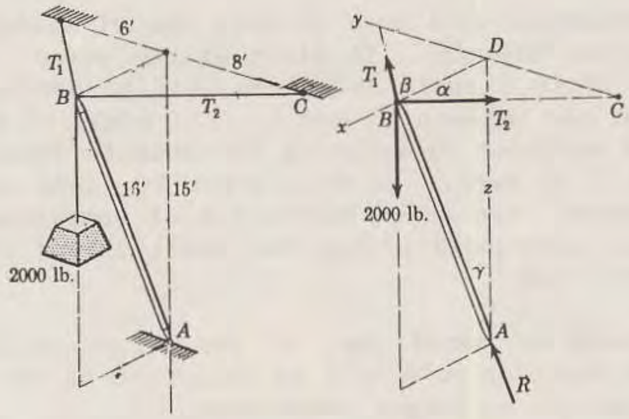
Most three-dimensional problems are simplified if they are broken down into three related two-dimensional problems. Projection of the forces on the three coordinate planes results in the three two dimensional systems involving the force components in the respective planes.

When equilibrium is produced by concurrent forces, the origin of coordinates is conveniently chosen at the point of concurrency. Since the moments of all forces are zero about any axis through this point, it follows that the equilibrium of concurrent forces is specified completely by the first three of Eqs. 16.

In the case of equilibrium produced by parallel forces, again only three conditions must be satisfied. If the forces are all perpendicular to the x-y plane, for example, then $\Sigma Fz = 0$. The moments of the forces about the other two axes must also be zero, so that $\Sigma Mx = 0$ and $\Sigma My = 0$.

SAMPLE PROBLEMS

1. Determine the compression R in the boom and the tensions T_1 and T_2 in the supporting cables if the weight of the boom is neglected compared with the applied load.



Solution: A space view of the free-body diagram of the boom is shown. For equilibrium the compression R must pass through B, the point of concurrency of the other three forces, and is therefore in the direction of the boom axis. From the geometry of the figure the following values, needed in the computation, are obtained:

$$BD = 5.57 \text{ ft.}$$

$$\cos\gamma = 0.938, \quad \sin\gamma = 0.348,$$

$$\cos\beta = 0.680, \quad \sin\beta = 0.733,$$

$$\cos\alpha = 0.571, \quad \sin\alpha = 0.821.$$

It may be observed that T_1 and T_2 lie in the x-y plane, and, hence, R may be computed by a force summation in the z-direction. Thus:

$$[\Sigma Fz = 0] : \quad R\cos\gamma = 2000, \quad R = 2130 \text{ lb.} \quad \underline{\text{Ans.}}$$

Equating to zero the moments about an axis through A parallel to the y-axis eliminates R from the equation and gives:

$$[\Sigma MAy = 0] : (T_1 \cos\beta)\, 15 + (T_2 \cos\alpha)\, 15 - 2000 \times 16 \sin\gamma = 0,$$

$$\text{or } 0.68\, T_1 + 0.571\, T_2 = 742$$

Equating to zero the forces in the y-direction gives

$$[\Sigma Fy = 0] : \quad T_1 \sin\beta - T_2 \sin\alpha = 0, \; 0.733\, T_1 = 0.821\, T_2.$$

Solution of the two equations yields:

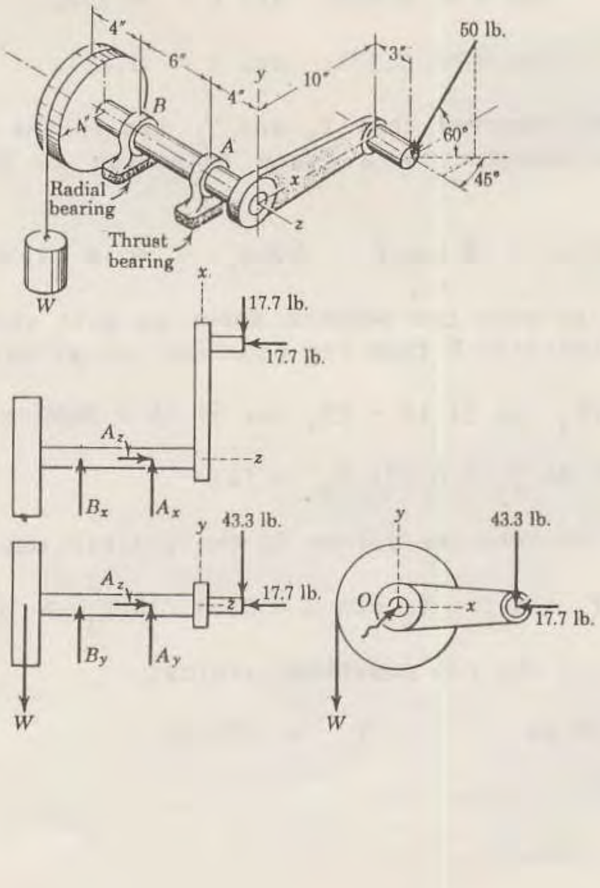
$$T_1 = 624 \text{ lb.}, \quad T_2 = 557 \text{ lb.} \quad \underline{\text{Ans.}}$$

The procedure used here is only one of several possible ones for obtaining the answers. In place of the moment equation about A the principle $\Sigma Fx = 0$ could have been used to obtain one of the two necessary relations between T_1 and T_2. It would be possible to avoid a simultaneous solution by equating the moments about a line passing through A and C to zero. In this event T_1 could be found from one equation. However, the added difficulty of computing moment arms to such an inclined axis would offset the small labour of a simultaneous solution in this case.

It should be noted that if the weight of the boom is not negligible the reaction at A will no longer be in the direction of AB since the forces are no longer concurrent.

---

2. A 50 lb. force is applied to the handle of the hoist in the direction shown. The bearing A supports the thrust (force in the direction of the shaft axis) while bearing B supports only radial load (load normal to the shaft axis). Determine the weight W which can be supported and the total radial force exerted on the shaft by each bearing.



Solution: The free-body diagram of the shaft, lever, and drum considered as a single body could be shown by a space view if desired but is represented here by its three orthogonal projections. The 50 lb. applied force is resolved into its three components, and each of the three views shows two of these components. The correct directions of Ax and Bx may be seen by inspection if it is observed that the resultant of the two 17.7 lb. forces exerts a counterclockwise moment about A. The correct directions of Ay and By can not be determined until the magnitudes of the moments are obtained, so these forces may be arbitrarily assigned. The x-y projection of the bearing forces is shown by a wavy arrow since its direction is unknown. The addition of Az and W completes the free-body diagrams. It should be noted that the three views represent three two-dimensional problems related by the corresponding components of the forces.

From the x-y projection:

$[\Sigma M_o = 0]$: $4W - 10 \times 43.3 = 0, \quad W = 108.3$ lb. Ans.

From the x-z projection:

$[\Sigma M_A = 0]$: $6B_x + (7 \times 17.7) - (10 \times 17.7) = 0, \quad B_x = 8.85$ lb.,

$[\Sigma F_x = 0]$: $A_x + 8.85 - 17.7 = 0, \quad A_x = 8.85$ lb.

The y-z view gives:

$[\Sigma M_A = 0]$: $6 B_y + (7 \times 43.3) - (10 \times 108.3) = 0, \quad B_y = 130$ lb.,

$[\Sigma F_y = 0]$: $A_y + 130 - 43.3 - 108.3 = 0, \quad A_y = 21.6$ lb.,

$[\Sigma F_z = 0]$: $A_z = 17.7$ lb.

The total radial forces on the bearings become:

$[A_r = \sqrt{A_x^2 + A_y^2}$; $A_r = \sqrt{(8.85)^2 + (21.6)^2} = 23.3$ lb. Ans.

$[B = \sqrt{B_x^2 + B_y^2}$; $B = \sqrt{(8.85)^2 + (130)^2} = 130.2$ lb. Ans.

PROBLEMS

1. The bracket is secured to the end of the fixed shaft by a single bolt which produces sufficient force to hold it in place. Determine the resultant force and moment exerted by the bolt and shaft on the bracket. Ans. F = 800 lb., M = 9330 lb.in.

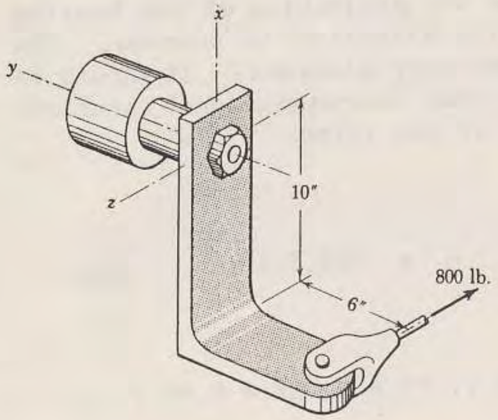


Fig. 1

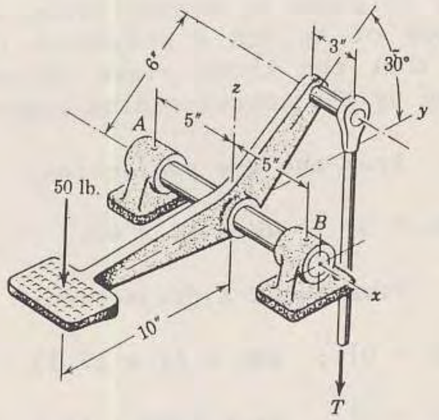


Fig. 2

2. A 50 lb. vertical force on the foot pedal of the bell crank is required to produce the tension T in the vertical control rod. Determine the bearing reactions at A and B.

Ans. A = 44.2 lb., B = 102 lb.

3. The control lever must supply a torque of 400 lb.in. about the axis O-O to turn the shaft. If the lever and the shaft are in contact at A, find the force F which produces the torque, and determine the total reaction exerted on the lever by the end of the shaft at A.

<u>Ans</u>. Reaction is 57.7 lb. force and 462 lb.in. couple

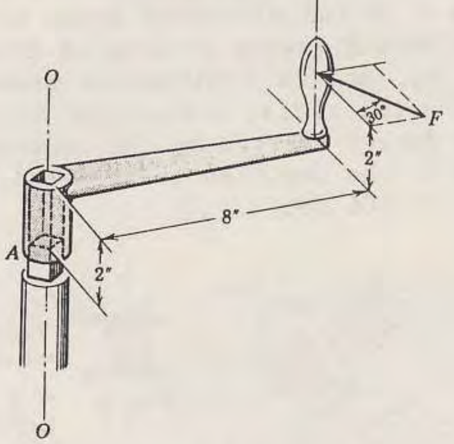


Fig. 3

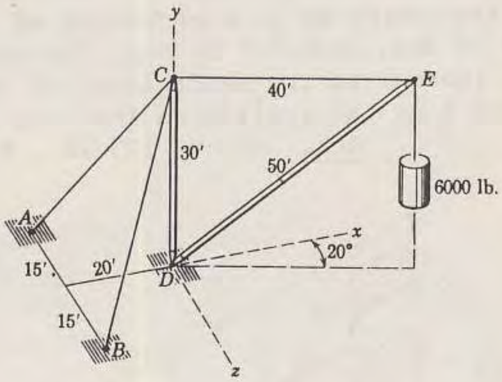


Fig. 4

4. Find the larger of the two tensions in the cables AC and BC for the boom hoist in the position shown. Neglect the weight of the boom. T = 10920 lb., T = 3780 lb. <u>Ans</u>.

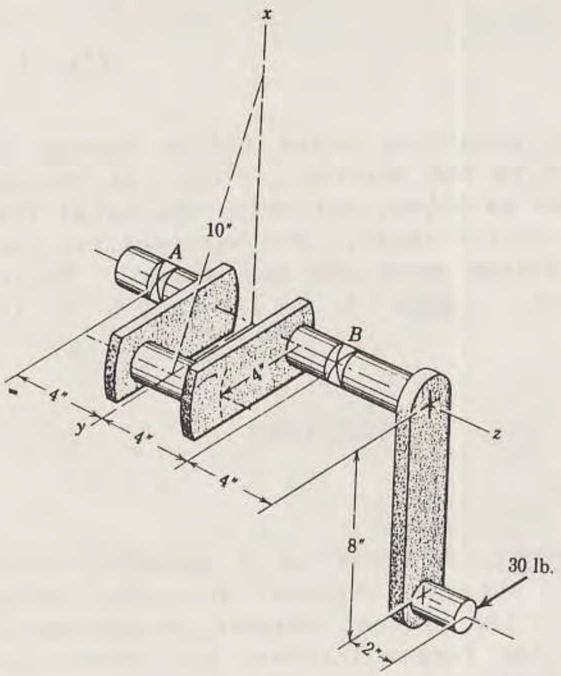


Fig.5

5. Determine the forces on the bearings A and B if the 30 lb. force is required to turn the crankshaft of the single-cylinder engine against the compression in the cylinder for the position shown. The axis of the 10 in. connecting rod is shown by the dotted line.

6. A steel shaft is mounted in a lathe between centers A and B and is driven from the face plate C in the direction shown by the tang of the clamped dog D. The lathe tool E exerts a force of 200 lb. on the shaft at a mean radius of 2 in. and in a direction given by β = 70 deg. and α = 10 deg. Determine the radial components Ar. and Br. (normal to the shaft axis) of the forces exerted by the centers at A and B on the shaft when the tang is in the position illustrated.
Ans. Aγ = 123 lb., Bγ = 106 lb.

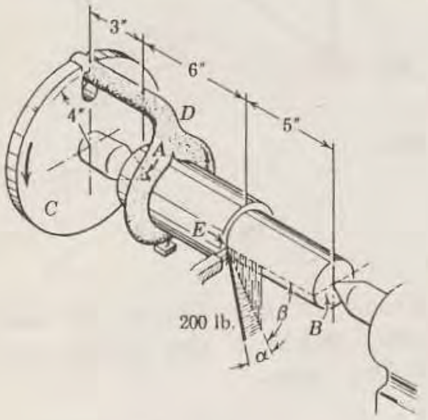


Fig. 6

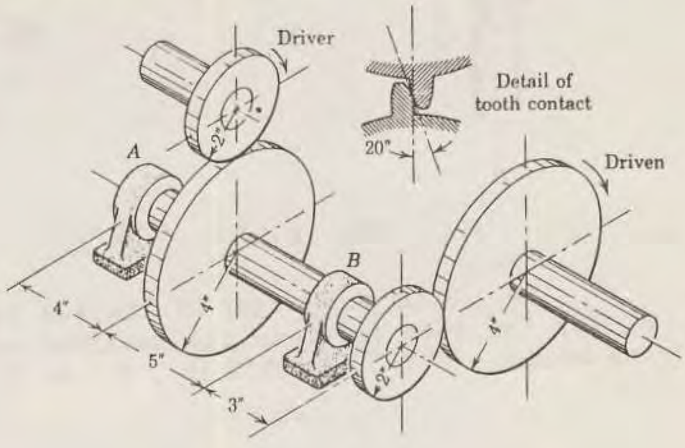


Fig. 7

7. The 4:1 reduction drive has a torque of 1000 lb.in. supplied to the shaft of the driving pinion. If the tooth action for both pairs of gears is as shown, determine the total forces exerted by the bearings A and B on the shaft. The bearings for the shafts of the driving pinion and driven gear are not shown. The weights of the parts may be neglected. Ans. A = 279 lb., B = 1580 lb.

STRUCTURES

18. **Structures:** A truss is a structure that consists of three or more members attached together at points called "joints" to form a rigid body. In the present chapter, attention is drawn toward the determination of the forces internal to a structure. The joints are assumed to be frictionless pin joints that can not exert torques

on the members. Thus, only compressive or tensile forces act along members of the truss. In the force analysis of structures it is necessary to dismember the structure and to analyze separate free-body diagrams of individual members or combinations of members in order to determine the forces internal to the structure. This analysis calls for very careful observance of Newton's third law which states that each action is accompanied by an equal and opposite reaction.

Members, loads, and reactions lie in the same plane. The weight of each member is negligible compared to the loads. In diagrams the width of a member is ignored, and it is represented by a line, and the load and reactions act only at the joints.

19. **Simple Trusses:** The basic "triangular truss", as shown in Fig.30, introduces Bow's notation. Each region or field inside and outside the truss is assigned a letter. These fields are designated a, b, c, and d. The members between a and b, b and c, and b and d are ab, bc, and bd, respectively. The joints are designated 1, 2, and 3.

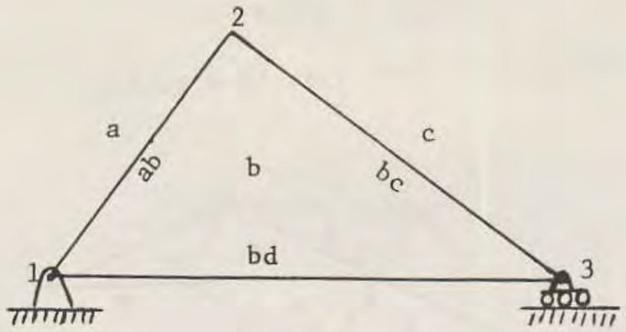


Fig.30

Alternatively, a joint is designated by naming clockwise the fields surrounding it. Thus, in Fig.30, joint 1 is joint abd., joint 2 is joint acb, and joint 3 is bcd.

Simple trusses are generated from the basic triangular truss by successively adding a pair of new members to existing joints and also, by connecting the two new members at a new joint. Thus, a simple truss will always have an odd number of members. When more members are present than are needed to prevent collapse, such additional members or supports which are not necessary for maintaining the equilibrium position are called "redundant".

SAMPLE PROBLEMS

1. Prove that if a simple truss has m members and j joints:

$$m = 2j - 3.$$

Solution: For the basic triangular truss, j = 3. A joint is added for each additional pair of members. Thus, if m-3 members are added to the original 3 members, totalling m members, an additional (m-3)/2 joints are added to the original 3 joints. Therefore:-

$$j = 3 + \frac{m-3}{2} = \frac{m+3}{2}$$

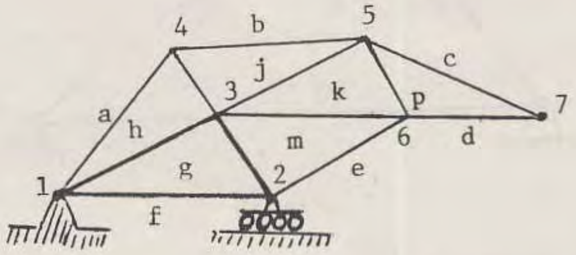
$$\text{or} \quad \underline{m = 2j - 3.} \qquad \text{Q.E.D.}$$

2. Show that the truss of the figure below is not a simple truss.

Solution: The number of members and joints of the truss are:

$$m = 12, \qquad j = 7$$

$$\text{clearly, } m \neq 2j-3;\ \text{i.e. } 12 \neq 14-3.$$



Alternatively, beginning with members gf, hg, and gm as the basic triangular truss and then adding successive pairs of members me and km, jk and kp, cp and pd, bj and hj, then a single member, i.e., member ah, must be added to complete the truss. This violates the definition of a simple truss.

3. In the figure of problem no.2, let the field g consisting of members hg, gf, and gm be the basic triangular truss. Find the members that may be removed to reduce the truss to a simple truss.

Solution: Note that any one of members me, km, jk, bj, kp, ah and hj may be removed to reduce the truss to a simple truss. Check the procedure as follows:

(a) Omit me; generate truss by successively adding the pairs ah and hj, jk and bj, km and kp, cp and pd.

(b) Omit km; generate truss by successively adding the pairs ah and hj, jk and bj, me and kp, cp and pd.

(c) Omit jk; generate truss by successively adding the pairs ah and hj, me and km, bj and kp, cp and pd.

(d) Omit bj; generate truss by successively adding the pairs ah and hj, me and km, jk and kp, cp and pd.

(e) Omit kp; generate truss by successively adding the pairs ah and hj, me and km, jk and bj, cp and pd.

(f) Omit ah; generate truss by successively adding the pairs me and km, jk and kp, hj and bj, cp and pd.

(g) Omit hj; generate truss by successively adding the pairs me and km, jk and kp, bj and ah, cp and pd.

20. **Design of simple Trusses:** The design of a truss involves the determination of the forces in the various members and the selection of appropriate sizes and structural shapes to withstand the forces. There are several assumptions made in the force analysis of simple trusses. First, all members are assumed to be two-force members. A two-force member is one in equilibrium under the action of two-forces only. These two forces are applied at the ends of the member and are necessarily equal, opposite, and collinear for equilibrium. The member may be in tension or compression, as shown in Fig.31. It should be noted that in representing the equilibrium of a portion of a 2-force member the tension T or compression C acting on the cut section is the same for all sections. It is assumed here that the weight of the member is small compared with the force it supports. If such is not the case or if the small effect of the weight is to be accounted for, the weight W of the member, if uniform, may be assumed to be replaced by two forces, each $\frac{W}{2}$, acting at each end of the member. These forces, in effect, are treated as loads externally applied to the pin connections. Accounting for the weight of a member in this way gives the correct result for the average tension or compression along the member but will not account for the effect of bending of the member.

When riveted or welded connections are used to join structural members, the assumption of a pin-jointed connection is usually satisfactory if the centerlines of the members are concurrent at the joint as in Fig.31.

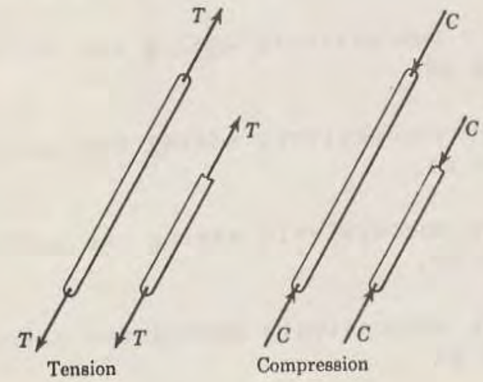


Fig. 31

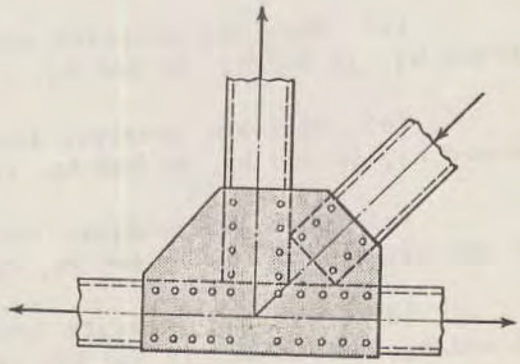


Fig. 32

Furthermore, it is assumed in the analysis of simple trusses that all external forces are applied at the pin connections. This condition is satisfied in most trusses. In bridge trusses the deck is usually laid on cross beams that are supported at the joints. Provision for expansion and contraction due to temperature changes and for deformations resulting from applied loads is usually made at one of the supports for large trusses. A roller, rocker, or some kind of slip joint is provided. Trusses and frames wherein such provision is not made are statically indeterminate.

There are three methods for the force analysis of simple trusses. All references will be made to the simple truss shown in Fig.33 for each of the three methods. The free body diagram of the truss as a whole is shown in Fig. 33b., and for this example a system of Bow's notation is used as before.

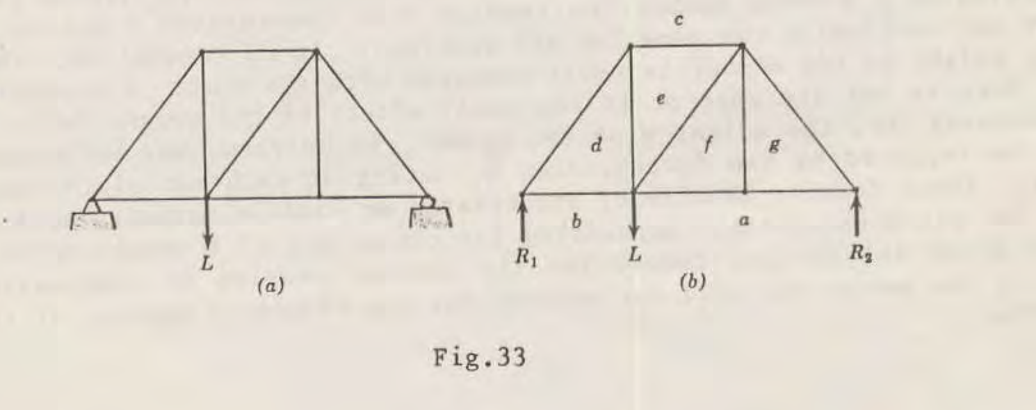


Fig.33

21. **Method of Joints:** Although any convenient notation may be used to designate each member and joint, such as the labelling of each joint by a letter or number, Bow's notation is employed here. Each joint is uniquely designated by the clockwise sequence of letters surrounding the joint. Thus the pin at the right support, Fig.33b, is designated as joint "agc", and the joint at which the load L is applied is denoted by "abdef". The method of joints is a procedure for determining the compressive or tensile forces in the members of a truss. The joints of a truss are in equilibrium. A free-body diagram is drawn for each joint. The forces in all members as well as the reactions at the truss-support points are then determined by solving the equilibrium equations written for each joint. Determine the unknown by proceeding from one joint to another with the aid of a free-body diagram of the joint. The method therefore deals with the equilibrium of concurrent forces, and only two independent equilibrium equations are involved. The analysis is begun with any joint where at least one known load exists and where not more than two unknown forces are present. Solution may be started with the pin at the left end, and its free-body diagram is shown in Fig.34. The proper directions of the forces should be evident for this simple case by inspection. The free-body diagrams of portions of members cd and db are also shown to indicate clearly the mechanism of the action and reaction. The member db actually makes contact on the left side of the pin, although the force DB is drawn from the right side and is shown

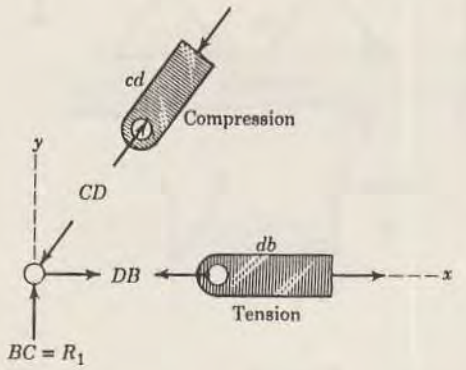


Fig.34

acting away from the pin. Thus, if the force arrows are consistently drawn on the same side of the pin as the member, then tension (such as DB) will always be indicated by an arrow away from the pin, and compression (such as CD) will always be indicated by an arrow toward the pin. The magnitude of CD is obtained from the equation $\Sigma Fy = 0$, and DB is then found from $\Sigma Fz = 0$.

Joint dce must be analyzed next since it now contains only two unknowns CE and ED. Joints bdefa, afg, cgfe, and agc are subsequently analyzed in that order. The free-body diagram of each joint and its corresponding force polygon which represents graphically the two equilibrium conditions $\Sigma Fx = 0$ and $\Sigma Fy = 0$ are shown in Fig.35. The numbers indicate the order in which the joints are analyzed. The force polygons shown have a particular significance which will be described in the next article. It should be noted that, when joint agc is finally reached, the computed reaction R_2 must be in equilibrium with

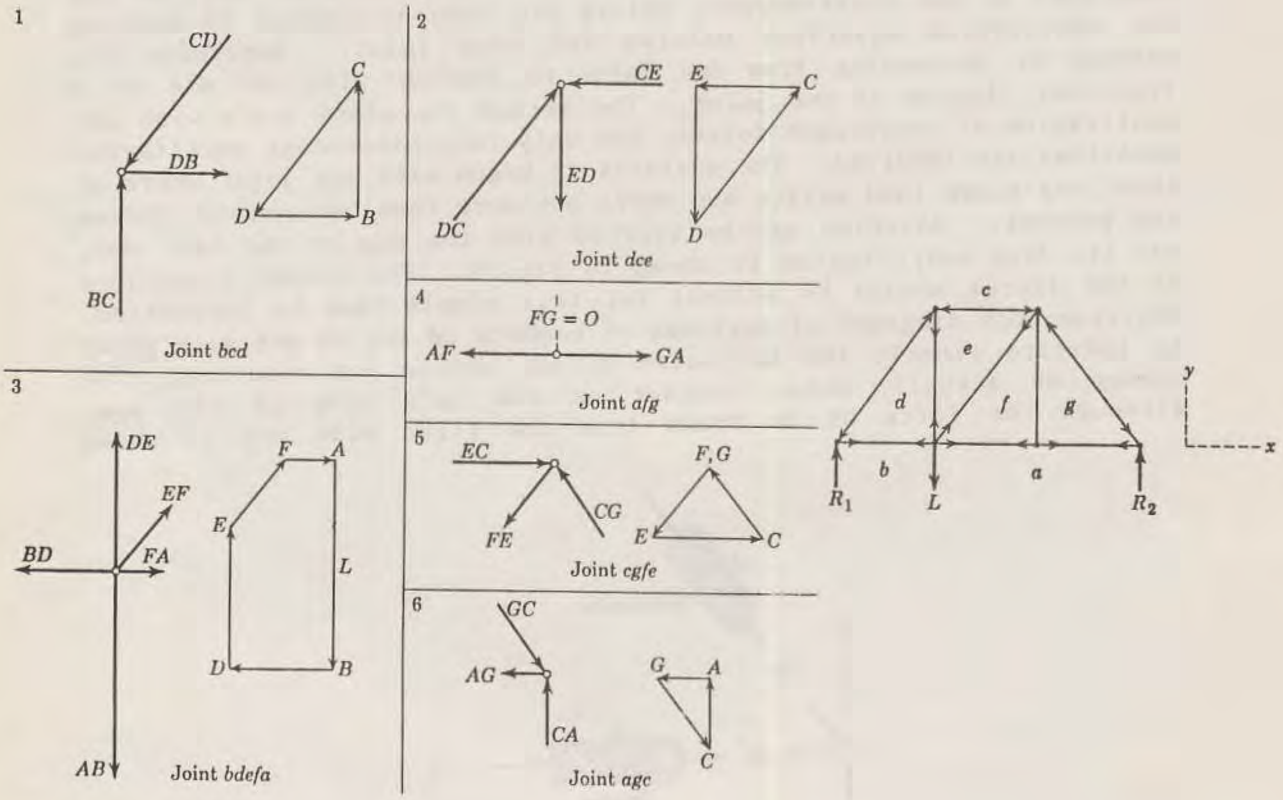


Fig.35

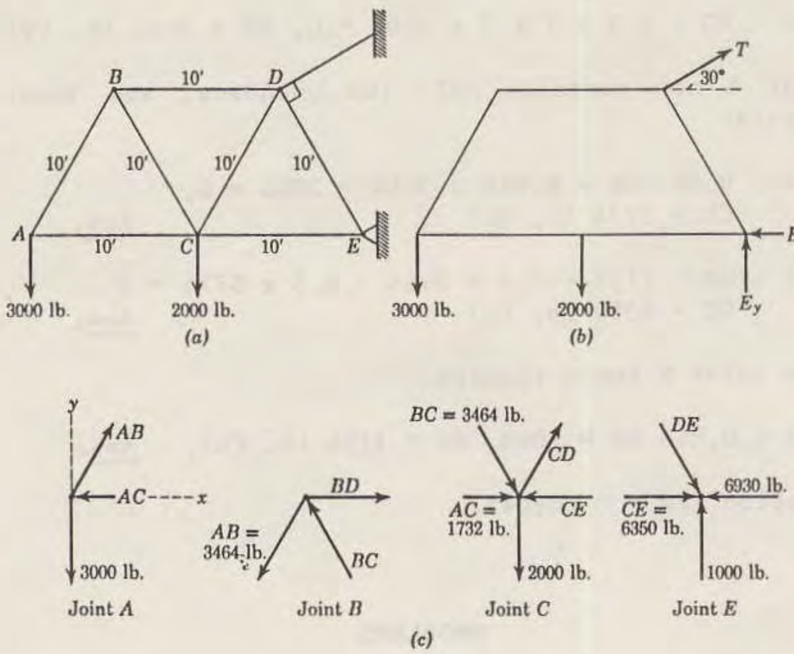
the forces in members ag and gc, determined previously from the two neighboring joints. This requirement will provide a check on the correctness of the work. It should also be noted that isolation of joint afg quickly discloses the fact that the force in fg is zero when the equation $\Sigma Fy = 0$ is applied. The force in this members would not be zero, of course, if an external load were applied at afg.

It is often convenient to indicate the tension T and compressions C of the various members directly on the original truss diagram by drawing arrows away from the pins for tension and toward the pins for compression. This designation is illustrated at the bottom of

Fig.35. In some instances it is not possible to assign initially the correct direction of one or both of the unknown forces acting on a given pin. In this event an arbitrary assignment may be made. A negative value from the computation indicates that the assumed direction should be reversed.

SAMPLE PROBLEM

1. Compute the force in each member of the loaded cantilever truss.



Solution: With joints indicated by letters the force in each member will be designated by the two letters defining the ends of the member. If it were not desired to calculate the external reactions at D and E, the analysis for a cantilever truss could begin with the joint at the loaded end. However, this truss will be solved completely, so that the first step will be to compute the external forces at D and E from the free-body diagram of the truss as a whole in part (b) of the figure. The equations of equilibrium give:-

$$T = 8000 \text{ lb.}, \quad Ex = 6930 \text{ lb.}, \quad Ey = 1000 \text{ lb.}$$

In part (c) of the figure are drawn the free-body diagrams showing the forces acting on each of the connecting pins. The correctness of the assigned directions of forces is verified when each joint is considered in sequence. There should be no question as to the correct direction of the forces on joint A. Equilibrium requires:

[ΣFy = 0] : 0.866 AB - 3000 = 0, AB = 3464 lb. (T) Ans.

[ΣFx = 0] : AC - 0.5 x 3464 = 0, AC = 1732 lb. (C) Ans.

Joint B must be analyzed next since there are more than two unknown forces on joint C. The force BC must provide an upward component in which case BD must balance the force to the left. Again the forces are obtained from:

[ΣFy = 0] : 0.866 BC - 0.866 x 3464 = 0, BC = 3464 lb. (C) Ans.

[ΣFx = 0] : BD - 0.5 x 2 x 3 x 3464 = 0, BD = 3464 lb. (T) Ans.

Joint C now contains only two unknowns, and these are found as before:

[ΣFy = 0] : 0.866 CD - 0.866 x 3464 - 2000 = 0,
CD = 5774 lb. (I) Ans.

[ΣFx = 0] : CE - 1732 - 0.5 x 3464 - 0.5 x 5774 = 0
CE = 6350 lb. (C) Ans.

Lastly, from joint E there results

[ΣFy = 0] : 0.866 DE = 1000, DE = 1154 lb. (C), Ans.

and the equation ΣFx = 0 checks.

PROBLEMS

1. Determine the force in each member of the truss.
Ans. AB = BC = 1000 lb.C.; AD = DC = 866 lb. T.;
BD = 1000 lb.T.

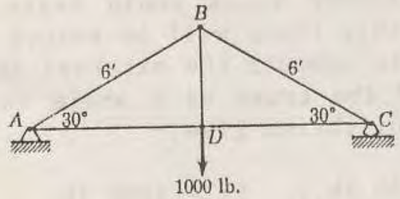


Fig. 1

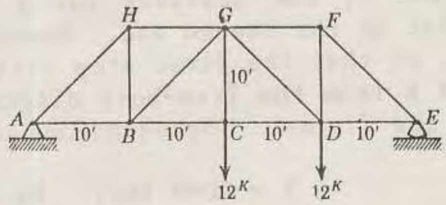


Fig. 2

2. Determine the force in members CD and DG of the Howe truss. Ans. CD = 18 kips T, DG = 4.24 kips C.

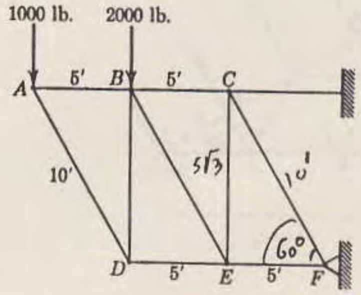


Fig. 3

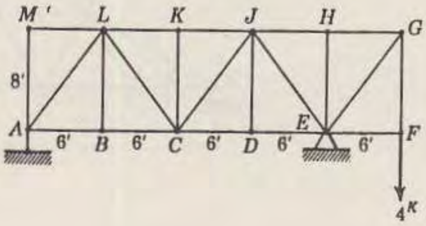


Fig. 4

3. Determine the force in members BE and EF of the cantilever truss. Ans. EF = 2308 lb.C, BE = 3464 lb.C.

4. Determine the force in members DE and JE for the deck truss shown.

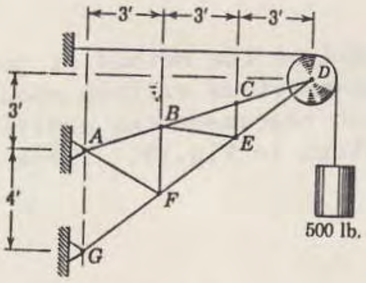


Fig. 5

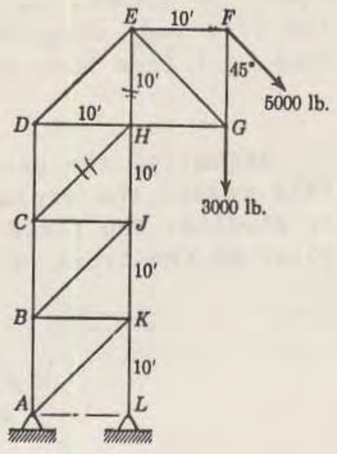


Fig. 6

5. Calculate the force in members BE, CE, and FE for the small hoisting truss. Ans. BE = CE = 0, FE = 950 lb.C.

6. Calculate the force in members BK and GE for the tower truss. Ans. BK = 3540 lb.C.; GE = 9240 lb.T.

7. Find the force in members EF, KL, and GL for the Fink truss shown. (Hint: Note that the forces in BP, PC, DN, etc. are zero)
Ans. EF = 40.4 kips C., KL = 20.0 kips T.,
GL = 10.0 kips T.

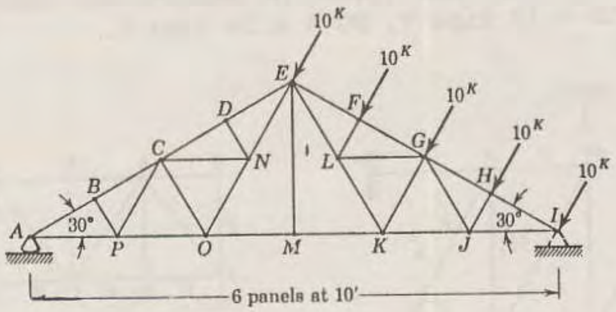


Fig.7

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22. **Graphical Method**: A graphical solution by the use of "Maxwell Diagrams" constitutes one method to solve truss problems. At each joint, there can be no more than two unknown forces acting and for each pair of joints, only one connecting member. The connection between the free body diagrams of the two joints is the force in the member. Thus, all free body diagrams are incorporated into the Maxwell Diagram.

Actually, the procedure is based on the method of joints, and for this reason the explanation in the previous article should be thoroughly studied. The force polygons which represent the equilibrium of each joint of the truss of Fig.33 are shown in Fig.35. These poly-

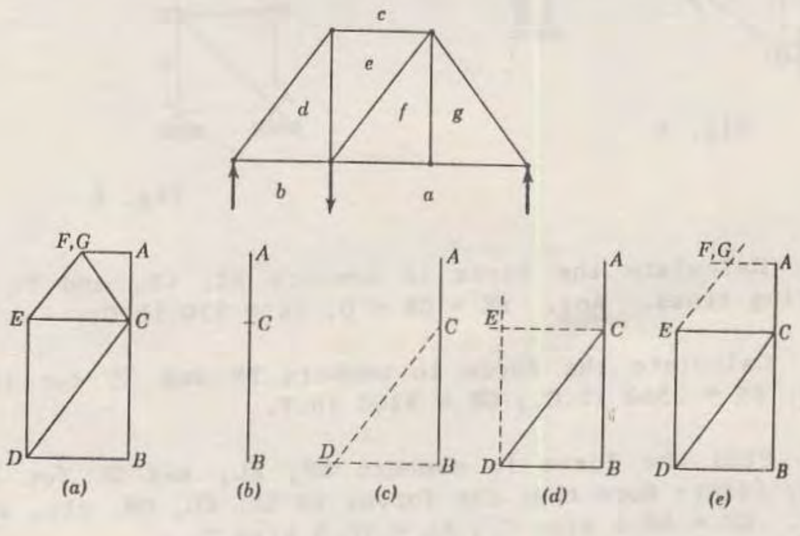


Fig.36

gons have been formed by vector addition of the forces consistently taken in a clockwise order around each respective joint. This consistent order makes it possible to superimpose all the polygons on a single diagram as shown in Fig.36a. The resulting figure is called the Maxwell Diagram for this truss, and it contains all the force polygons for each joint. The arrow heads have been omitted since the direction of the force in a member will depend on which of two joints is considered.

The Maxwell Diagram may be constructed directly from the original truss without the necessity for prior calculation of the forces in the internal members. The diagram requires again the use of Bow's notation, and a clockwise order for labeling the external forces on the truss is generally used. As a first step the external reactions are determined usually by calculation. Next the equilibrium polygon of forces external to the truss as a whole is drawn. If the loads and reactions are all parallel, this polygon becomes a line, shown in Fig.36b. for this truss.

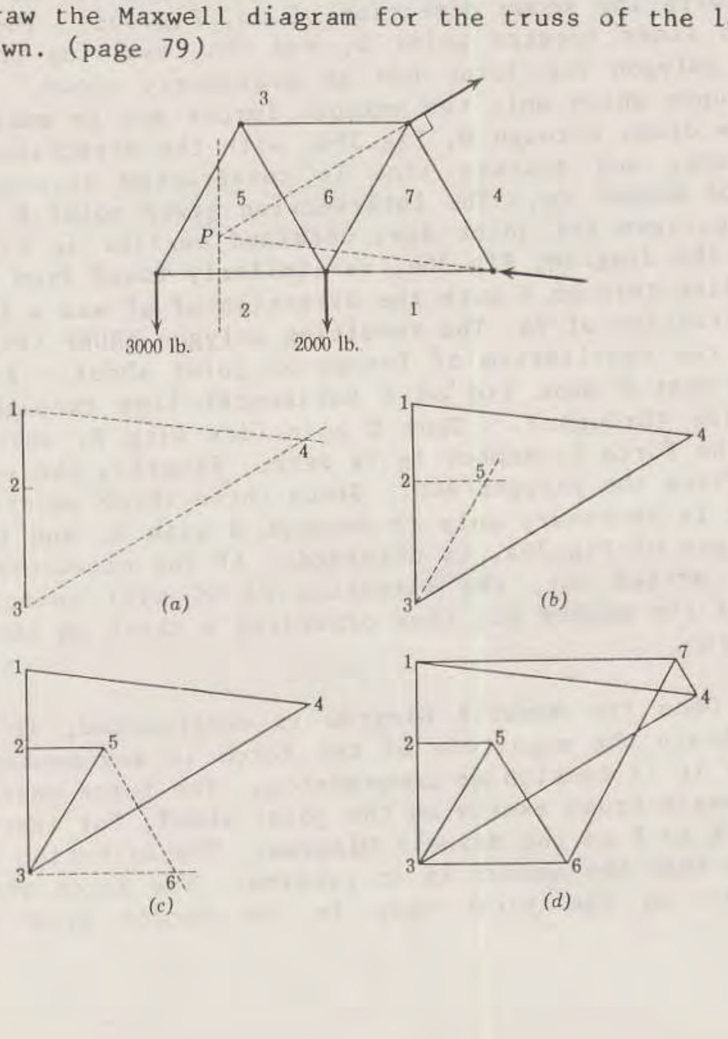
The force polygon of a joint where only two unknown forces act, such as bcd, is first drawn. This is accomplished by merely constructing a line through B with the known direction of db and a line through C with the known direction of cd, Fig.36c. The intersection of the two lines locates point D, and the resulting triangle BCD is the force polygon for joint bcd as previously shown. The adjacent joint dce upon which only two unknown forces act is analyzed next. A line is now drawn through D, Fig.36d, with the direction of member de on the truss, and another line is constructed through C with the direction of member ce. The intersection gives point E and completes the force polygon for joint dce, obtained earlier in Fig.35, step 2. Point F on the diagram, Fig.36e, is similarly found from the intersection of a line through E with the direction of ef and a line through A with the direction of fa. The resulting polygon ABDEF (Fig.35, step 3) represents the equilibrium of forces on joint abdef. From joint afg it is seen that G must lie on a horizontal line through A and on a vertical line through F. Thus G coincides with F, which checks the fact that the force in member fg is zero. Finally, the equilibrium of joint agc gives the polygon AGC. Since these three points are already obtained it is necessary only to connect G with C, and thus the completed polygon of Fig.36a. is obtained. If the construction has been accurately carried out, the direction of GC will coincide with the direction of the member gc, thus providing a check on the correctness of the diagram.

Once the Maxwell Diagram is constructed, it is a simple matter to obtain the magnitude of the force in any member and determine whether it is tension or compression. The force which the member ef of the sample truss exerts on the joint abdef, for instance, is the vector from E to F on the Maxwell Diagram. The direction is away from the joint so that the member is in tension. The force which the same member exerts on the joint cgfe is the vector from F to E and likewise indicates tension since it is away from the joint. It should be carefully noted that the proper sequence of letters for the force vector is obtained from a clockwise travel about the joint in question. A counterclockwise diagram could, of course, be used with equal success, provided this direction was consistently used in the notation and the interpretation. Such a diagram would result in an upside-down mirror image of the clockwise construction but is not used in practice.

Although the load line (or polygon) of external forces is normally the first step in the construction of the Maxwell diagram, in some cases it may be just as convenient or even necessary to begin the construction without the initial determination of the external reactions.

SAMPLE PROBLEM

1. Draw the Maxwell diagram for the truss of the last sample problem as shown. (page 79)



Solution: Bow's notation with numbers is used to avoid confusion with the letters previously employed to designate each joint. The external reactions may be obtained graphically from the concurrency principal for three forces by locating the intersection P of the line of action of the force in member 3-4 with the line of the resultant of the two applied loads. The reaction 4-1 must pass through this point. The force polygon representing the equilibrium of the truss as a whole is next drawn as shown in part (a).

The diagram is constructed on the force polygon. By starting with the left end of the truss, point 5 is seen to lie on a horizontal line through point 2 and on a line through point 3 having the direction of 3-5 as illustrated in part (b) of the drawing. The triangle 2-3-5 represents in that order the force polygon for joint 2-3-5. The same three forces are described in the figure for the last sample problem for joint A. Point 6 is located in similar manner in part (c) of the figure. Point 7 may be obtained from joint 1-7-4 and the diagram completed by joining points 6 and 7 as shown in part (d) of the illustration. The line joining points 6 and 7 is parallel to the member 6-7, which forms a check on the work.

In interpreting the diagram to obtain the force in member 6-5, for example, it is first necessary to consider one of the two joints to which this member is attached. If joint 3-6-5 is chosen, the polygon 3-6-5 is traced on the Maxwell diagram in that order, and the force which member 6-5 exerts on the pin is up and to the left (from 6 to 5). This is a compressive force since it is toward the pin. Scaling the diagram gives a magnitude for the line 6-5 in agreement with that already obtained. If joint 1-2-5-6-7 had been considered, the force would be designated 5-6 (clockwise order around the joint). The direction of the force is from 5 to 6 on the Maxwell diagram, which is toward the joint and is likewise compression.

PROBLEMS

1. Determine the force in each member of the truss with the aid of a free-hand sketch of the Maxwell diagram.

Ans. AE = EC = 2P, T; DE = EB = 2P, T; AB = $P\sqrt{3}$, C; BC = P, C

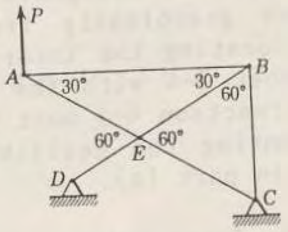


Fig. 1

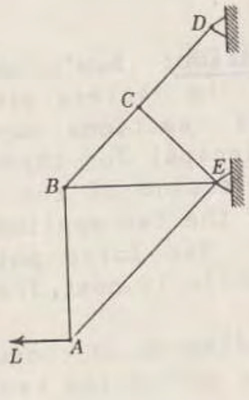


Fig. 2

2. Determine the force in members BE and BC from a freehand sketch of the Maxwell diagram. All angles are 45° or 90°.
<u>Ans.</u> BE = L, T; BC = L 2, C.

3. Determine the force in members JD and HD for the crane truss.

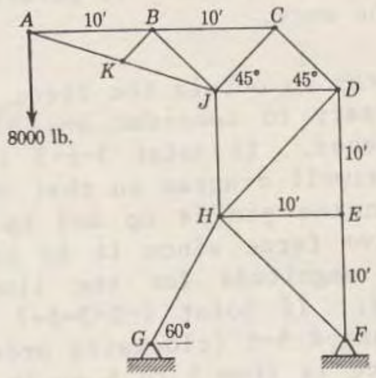


Fig. 3

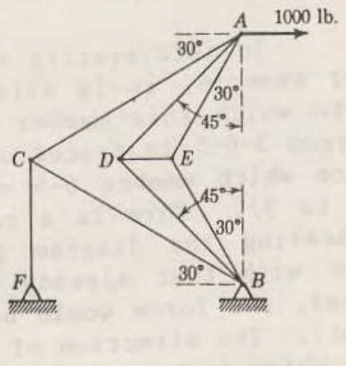


Fig. 4

4. Find the force in member DE for the loaded truss shown.

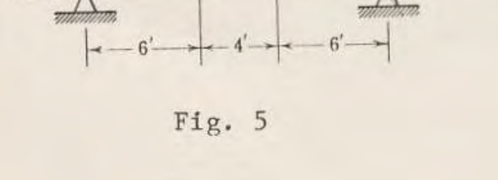


Fig. 5

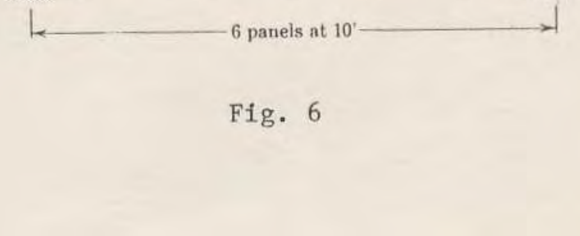


Fig. 6

5. The hinged frames ACE and DFB are connected by two hinged bars AB and CD which cross without being connected. Draw the Maxwell diagram and determine the force in AB. Ans. AB = 377 lb., C.

6. Draw the Maxwell diagram for the Fink truss loaded as shown. Find the force in member LG (Hint: For the right side of the truss the force in EF is unaffected by replacing members FL and GL by a member from F to K. The diagram can be completed with the temporary member and modified to account for the resubstitution of the original two members. By reason of symmetry only one-half of the diagram need be drawn). Ans.: LG - 4.33 kips T.

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23. **Method of Sections:** In the previous 2 methods, advantage is taken of only two of the three equilibrium equations since the procedures involve concurrent forces at each joint. The third equilibrium principle may be used to advantage by considering an entire section of the truss as a free-body in equilibrium under the action of a non-concurrent system of forces. This method of sections has the basic advantage that the force in almost any desired member may be found directly from an analysis of a section which has cut that member. Thus it is not necessary to proceed with the calculation from joint to joint until the member in question has been reached. In choosing a section of the truss it should be noted that in general not more than three members whose forces are unknown may be cut since there are only three available equilibrium relations which are independent.

The method of sections is now illustrated in the previous truss shown. The external reactions are first computed as before, considering the truss as a whole. Now let it be desired to determine the force in the member ef. An immaginary section, indicated by the dotted line, is passed through the truss cutting it into two parts, Fig.37b. This section has cut three members whose forces are initially unknown. In order that the portion of the truss on each side of the section will remain in equilibrium it is necessary to apply to each cut member the force which was exerted on it by the member cut away. These forces, either tensile or compressive, will always be in the direction of the respective members for simple trusses composed of two-force members. The left-hand section is in equilibrium under the action of the applied load L, the end reaction Rl, and the three forces exerted on the cut members by the right-hand section which has been removed. The forces may usually be drawn with their proper senses by a visual approximation of the equilibrium requirements. Thus in balancing the moments about point 1 the force CE is clearly to the left, which makes it compression since it acts toward the cut section

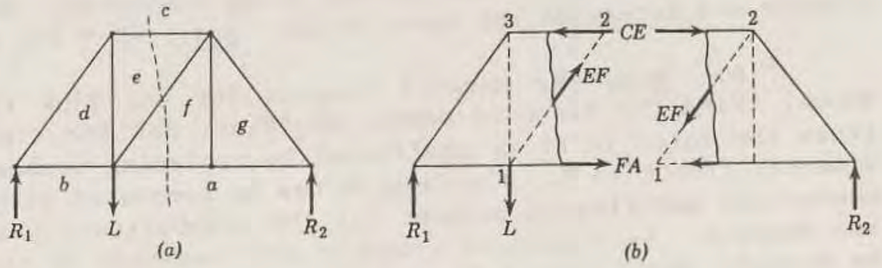


Fig.37

of member ce. The load L is greater than the reaction R1 so that the force EF must be up and to the right to supply the needed upward component for vertical equilibrium. Force EF is therefore tension since it acts away from the cut section. With the approximate magnitudes of R1 and L in mind the balance of moments about point 2 requires that FA be to the right. A casual glance at the truss should lead to the same conclusion when it is realized that the lower horizontal member will stretch under the tension caused by bending. The equation of moments about joint 1 eliminates three forces from the relation, and CE may be determined directly. The force EF is calculated from the equilibrium equation for the y-direction. Lastly FA may be determined by balancing moments about point 2. In this way each of the three unknowns has been determined independently of the other two.

The right-hand section of the truss, Fig.37b., is in equilibrium under the action of R_2 and the same three forces in the cut members applied in the directions opposite to those for the left section. The proper sense for the horizontal forces may easily be seen from the balance of moments about points 1 and 2.

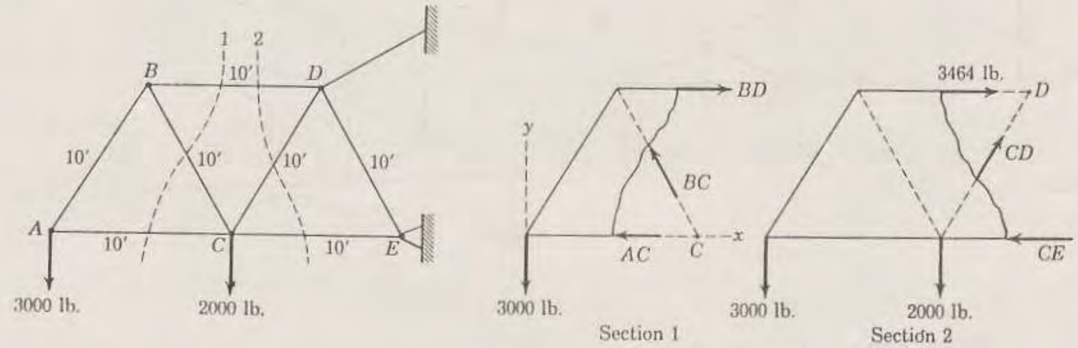
It is essential to understand that in the method of sections an entire portion of the truss is considered as a single body in equilibrium. Thus the forces in members internal to the section are not involved in the analysis of the section as a whole. In order to clarify the free body and the forces acting externally on it, the section is preferably passed through the members and not the joints.

Either section of a truss may be used for the calculations but the one involving the least number of forces will usually yield the simpler solution.

The moment equations may be used to great advantage in the method of sections, and a moment center through which as many forces pass as possible should be chosen. It is not always possible to assign an unknown force in the proper sense when the free body diagram of a section is drawn. With an arbitrary assignment made, a positive answer will verify the assumed sense and a negative result will indicate that the force is in the sense opposite to that assumed. Any system of notation desired may be used, although usually it is found convenient to letter the joints and designate a member and its force by the two letters defining the ends of the member.

SAMPLE PROBLEMS

1. Calculate the forces in members BC, BD, and CE for the truss shown.



Solution: Section 1 cuts only three members whose forces are unknown, and the free-body diagram of the portion of the truss to the left of the section is drawn. A force summation in the y-direction gives:

$[\Sigma F_y = 0]$: 0.866 BC - 3000 = 0, BC = 3464 lb.C. Ans.

Using C as a moment center gives:

$[\Sigma M_c = 0]$: 8.66 BD - 3000 x 10 = 0, BD = 3464 lb.T. Ans.

Section 2 may be used to find the force in CE. From the free-body diagram of the portion of the truss to the left of section 2 is obtained.

$[\Sigma M_D = 0]$: 8.66 CE - 2000 x 5 - 3000 x 15 = 0, CE = 6350 lb.C. Ans.

For a loaded cantilever truss, such as illustrated in this problem, it should be noted that the forces in the members may be obtained without calculation of the support reactions.

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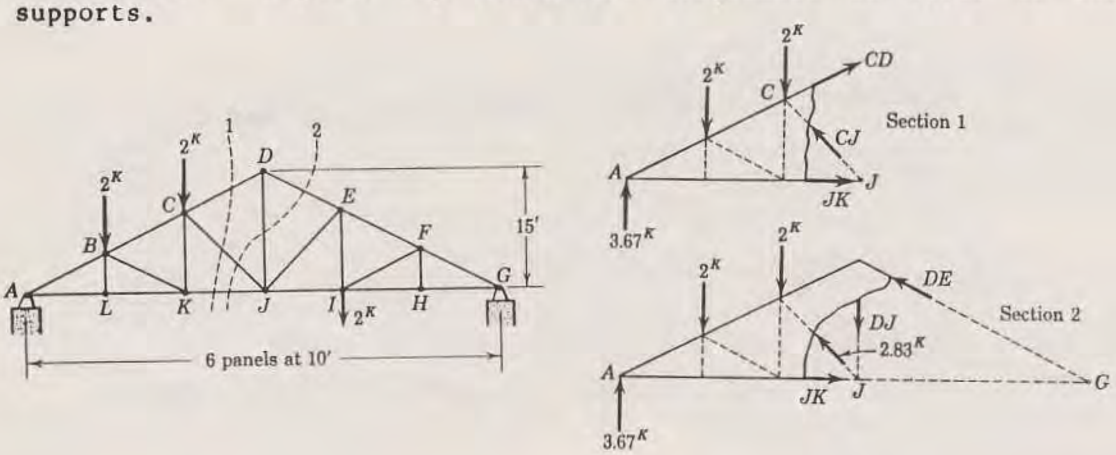
2. Calculate the force in member DJ of the Howe roof truss illustrated. Neglect any horizontal components of force at the supports.



Solution: It is not possible to pass a section through DJ without cutting four members whose forces are unknown. Although three of these cut by section 2 are concurrent at J and therefore the moment equation about J could be used to obtain the fourth, DE, the force in DJ can not be obtained from the remaining two equilibrium principles. It is necessary to consider first the adjacent section 1 before considering section 2.

The free-body diagram for section 1 is drawn and includes the reaction of 3.67 kips at A, which is previously calculated from the equilibrium of the truss as a whole. In assigning the proper directions for the forces acting on the three cut members a balance of moments about A eliminates the effects of CD and JK and clearly requires that CJ be up and to the left. A balance of moments about C eliminates the effect of the three-forces concurrent at C and indicates that JK must be to the right to supply sufficient counterclockwise moment. Again it should be fairly obvious that the lower chord is under tension due to the bending tendency of the truss. Although it should also be apparent that the top chord is under compression, the force in CD will be arbitrarily assigned as tension.

There is no harm in assigning one or more of the forces in the wrong direction as long as the calculations are consistent with the assumption. A negative answer will show the need for reversing the direction of the force. By the analysis of section 1, CJ is obtained from:

$[\Sigma MA = 0] : (0.707\ CJ)\ 30 - 2 \times 10 - 2 \times 20 = 0,\ CJ = 2.83$ kips, C.

In this equation the moment of CJ is calculated by considering its horizontal and vertical components acting at point J. Equilibrium of moments about J requires:

$[\Sigma MJ = 0] : (0.894\ CD)\ 15 + (3.67 \times 30) - (2 \times 10) - (2 \times 20) = 0,$
$CD = -3.73$ kips.

The moment of CD about J is calculated here by considering its two components as acting through D. The minus sign indicates that CD was assigned in the wrong direction. Thus:

$CD = 3.73$ kips, C. Ans.

If desired, the direction of CD may be changed on the free-body diagram and the algebraic sign of CD reversed in the calculations, or else the work may be left as it stands with a note stating the proper direction.

From the free-body diagram of section 2, which now includes the known value of CJ, a balance of moments about G is seen to eliminate DE and JK. Thus:

$[\Sigma MG = 0] : 30\ DJ + (2 \times 40) + (2 \times 50) - (3.67 \times 60) - (2.83 \times 0.707 \times 30) = 0,\ DJ = 3.33$ kips. T. Ans.

Again the moment of CJ is determined from its components considered as acting at J. The answer for DJ is positive so that the assumed tensile direction was correct. An analysis of the joint D alone also verifies this conclusion.

Although the procedure used here is undoubtedly the shortest for obtaining DJ, the student should consider other possibilities. It may be observed that, if for some other problem the 2 kip load at I were not present but the other loads remained the same, the forces in IE and JE (as well as in HF and IF) would be zero. In this event a section through members CD, DJ, JE and IJ would involve only three unknown forces, and a solution for DJ could be obtained with a single moment equation about A.

PROBLEMS

1. Calculate the force in members JH, JD, and CD for the Warren bridge truss. Ans. JH = 12.96 kips.C., JD = 2.92 kips.T., CD = 11.33 kips.T.

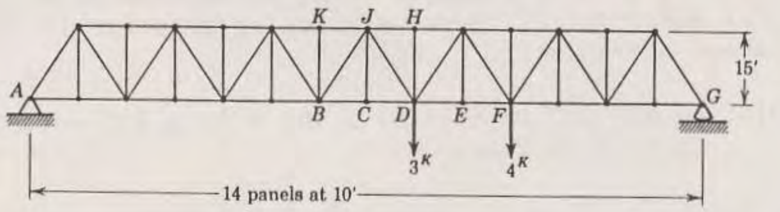


Fig.1

2. The truss shown is composed of 45 deg. right triangles. The crossed members in the center two panels are slender tie rods incapable of supporting compression. Retain the two rods which are under tension and compute the magnitude of their tensions. Also find the force in member MN.

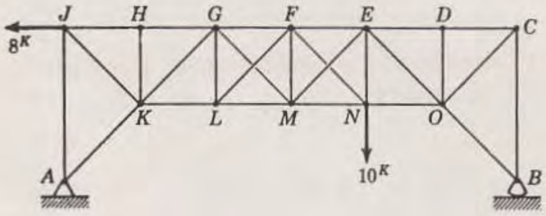


Fig. 2

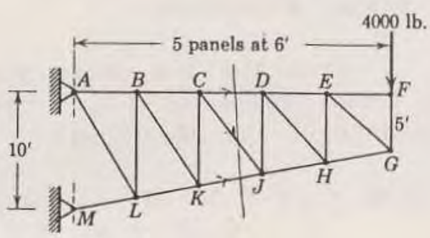


Fig. 3

3. Calculate the force in member CJ with only one free-body diagram and one equation of equilibrium.

4. Calculate the force in member CF of the loaded truss. All angles are 30°, 60° 90°, or 120°.

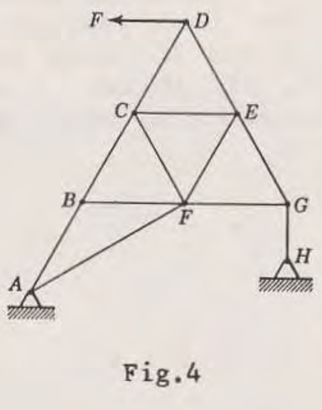


Fig.4

5. Determine the force in members DG, EG, FG, and MG for the cantilever crane truss. What can be said regarding the statical determinateness of the truss? Ans. DG = 5 kips.C, EG = 30 kips.C, FG = 11.55 kips.C, MG = 23.1 kips.C.

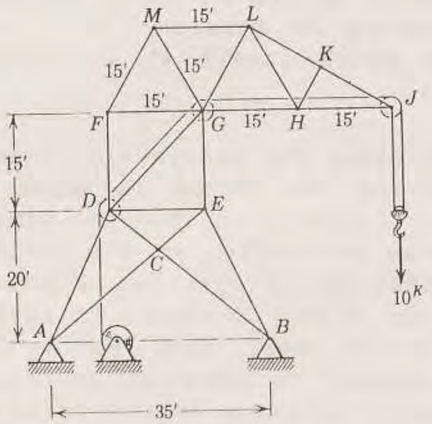


Fig. 5

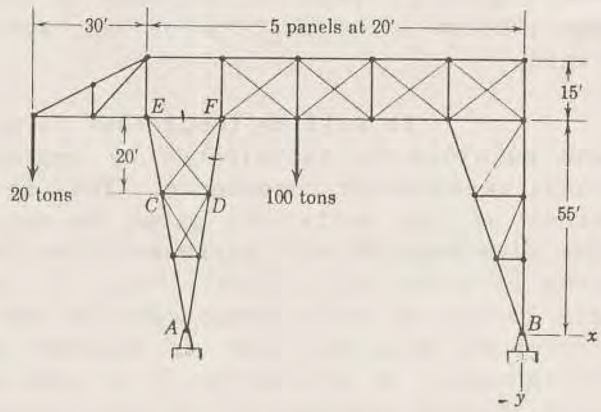


Fig. 6

6. In the travelling bridge crane shown all crossed members are slender tie rods incapable of supporting compression. Determine the force in members DF and EF and find the horizontal reaction on the truss at A. Ans. DF = 76.8 tons C, EF = 36.4 tons C, Ax = 10.1 tons

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24. **Frames and Machines:** Structures and mechanisms composed of joined members anyone of which has more than two forces acting on it can not be analyzed by the methods developed for simple trusses. Such members are multiforce members (three or more forces), and in general the forces will not be in the direction of the members. In the previous discussion the equilibrium of multiforce bodies was illustrated, but attention was stressed on the equilibrium of a single rigid body. Now it is focussing on the equilibrium of interconnected rigid bodies which contain multiforce members. The forces acting on each member of a connected system are found by isolating the member with a free-body diagram and applying the established equations of equilibrium. The principle of Action and Reaction must be carefully observed when representing the forces of interaction on the separate free-body diagrams. If the structure contains more members or supports than are necessary to prevent collapse, then, as in the case of trusses, the problem is statically indeterminate, and the principles of equilibrium although necessary are not sufficient for solution.

If the frame or machine constitutes a rigid unit by itself, the analysis is best begun by establishing all the forces external to the structure considered as a single rigid body. The structure is then dismembered and the equilibrium of each part is considered. The equilibrium equations for the several parts will be related through the terms involving the forces of interaction. If the structure is not rigid by itself but depends on its external supports for rigidity, then it is usually necessary to consider first the equilibrium of a portion of the system which itself is inherently rigid.

It will be found that in most cases the analysis of frames and machines is facilitated by representing the forces in terms of their rectangular components. This is particularly so when the dimensions of the parts are given in mutually perpendicular directions. The advantage of this representation is that the calculation of moment arms is accordingly simplified. It is not always possible to assign all forces or their components in the proper sense when drawing the free-body diagrams, and it becomes necessary to make an arbitrary assignment. In this event it is absolutely necessary that a force be consistently represented in the diagrams for interacting bodies which involve the force in question. Thus for two bodies which are connected

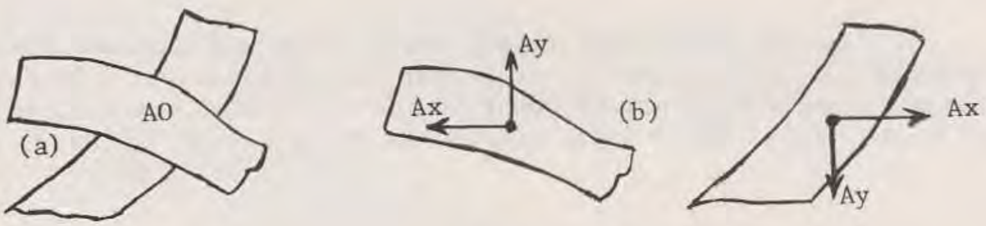


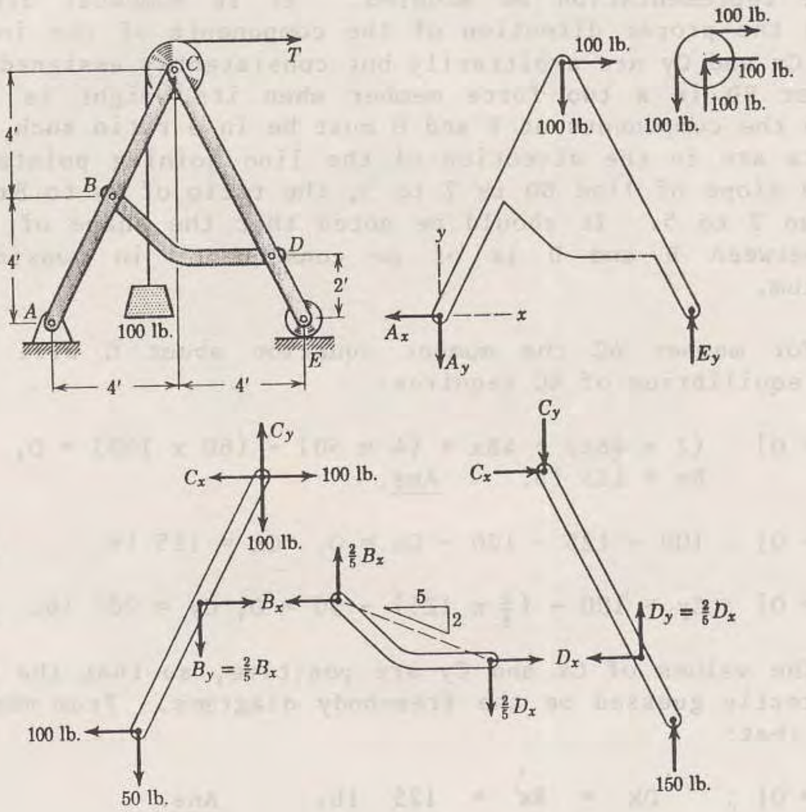
Fig.38

by the pin A, Fig.38a., when separated the components must be consistently represented in the opposite directions, Fig.38b. The assigned directions may prove to be wrong when the algebraic signs of the components are determined upon calculation. If Ax, for instance, should turn out to be negative, it is actually acting in the direction opposite to that originally represented. Accordingly it would be necessary to reverse the direction of the force on both members and to reverse the sign of this force term in the equations. Or the representation may be left as originally made, and the proper sense of the force will be understood from the negative sign.

Finally, situations occasionally arise where it is necessary to solve two or more equations simultaneously in order to separate the unknowns. In most instances, however, simultaneous solutions may be avoided by careful choice of the member or group of members for the free-body diagram and by a careful choice of moment centers which will eliminate undesired terms from the equations. The method of solution described is illustrated as follows:

SAMPLE PROBLEM

1. Determine the x and y components of all forces acting on each member of the hoisting frame. The weights of the parts are small compared with the load carried.



Solution: The frame is a noncollapsible unit if the external supports are removed, so the reactions external to the frame as a whole will be computed first. The free-body diagram of the entire frame minus the pulley is shown in the figure. The forces acting at C. are obtained from the free body of the pulley shown separately. The equations of equilibrium for the frame give:

$[\Sigma MA = 0]$: $8Ey - (4 \times 100) - (8x\ 100) = 0$, $Ey = 150$ lb.,

$[\Sigma Fy = 0]$: $Ay + 100 - 150 = 0$, $Ay = 50$ lb.,

$[\Sigma Fx = 0]$: $Ax = 100$ lb.

The members are now separated and all forces acting on each member are represented. The diagrams are best arranged in their approximate relative position to aid in designating the common forces of interaction. The forces exerted by the pulley on the frame can be considered as applied to either of the two members and are shown acting on AC in this solution. Or if desired a separate free-body diagram of the pulley shaft which connects the members could be drawn to account for the action of the applied loads and the reactions of the two members at C. It is suggested that the equivalence of this alternate representation be studied. It is somewhat difficult to visualize the proper direction of the components of the interactions at C, so Cx and Cy are arbitrarily but consistently assigned as shown. The member BD is a two force member when its weight is neglected. Therefore the components at B and D must be in a ratio such that their resultants are in the direction of the line joining points B and D. Since the slope of line BD is 2 to 5, the ratio of By to Bx and Dy to Dx is also 2 to 5. It should be noted that the shape of the actual member between B and D is of no consequence in considering its equilibrium.

For member AC the moment equation about C will yield Bx. Thus the equilibrium of AC requires:

$[\Sigma Mc = 0]$: $(2 \times \frac{2}{5}Bx) + 4Bx + (4 \times 50) - (80 \times 100) = 0$,
$Bx = 125$ lb. Ans.

$[\Sigma Fx = 0]$: $100 + 125 - 100 - Cx = 0$, $Cx = 125$ lb. Ans.

$[\Sigma Fy = 0]$: $Cy - 100 - (\frac{2}{5} \times 125) - 50 = 0$, $Cy = 200$ lb. Ans.

The values of Cx and Cy are positive, so that the directions were correctly guessed on the free-body diagrams. From member BD it is clear that:

$[\Sigma Fx = 0]$: $Dx = Bx = 125$ lb. Ans.

From which: $Dy = By = \frac{2}{5} \times 125 = 50$ lb. Ans.

All forces acting on CE have been found from the two other members. As a check on the correctness of the work the equilibrium of CE may be checked. Thus:

$[\Sigma Mc = 0]$: $(3 \times \frac{2}{5} \times 125) + (4 \times 150) - (125 \times 6) = 0$, (Check)

$[\Sigma Fx = 0]$: $125 - 125 = 0$ (Check)

$[\Sigma Fy = 0]$: $150 + (\frac{2}{5} \times 125) - 200 = 0$ (Check)

If desired the total reactions at each connection may be obtained by combining the two force components.

PROBLEMS

1. Determine the force F supported by the pin at A when the 100 lb. force acts on the linkage as shown. Ans. F = 269 lb.

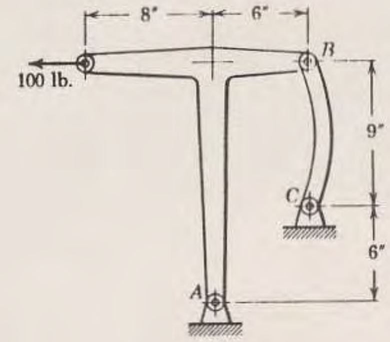


Fig. 1

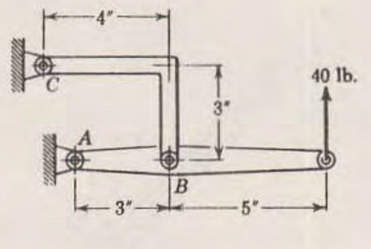


Fig. 2

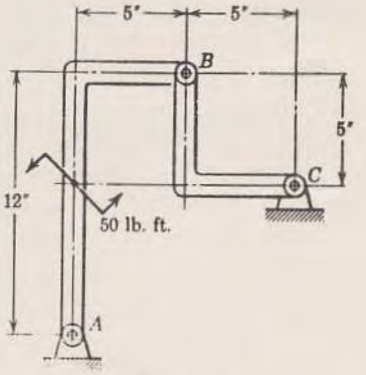


Fig. 3

2. Find the force supported by the pin at C. Ans. C = 178 lb.

3. Calculate the force supported by the pins at A and C for the connected links under the action of the applied couple.
Ans. A = C = 49.9 lb.

4. A dual-grip clamp shown in the figure is used to provide added clamping force with a positive action. If the vertical screw is tightened to produce a clamping force of 1000 lb. and then the horizontal screw is tightened until the force at A is doubled, find the total clamping force P and the reaction R on the pin at B.
Ans. P = 2000 lb., R = 2400 lb.

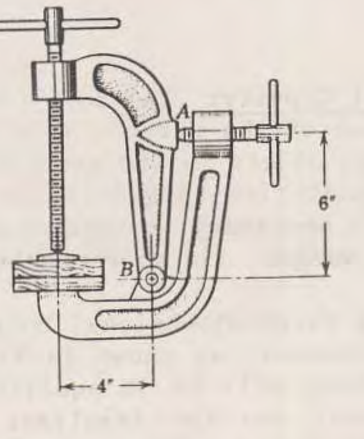


Fig.4

5. Determine the shearing force Q applied to the bar if a 50 lb. force is applied to the handle for 0 = 30 degrees. For a given applied force what value of 0 gives the greatest shear?

Ans. Q = 1835 lb., 0 = 0 for max Q.

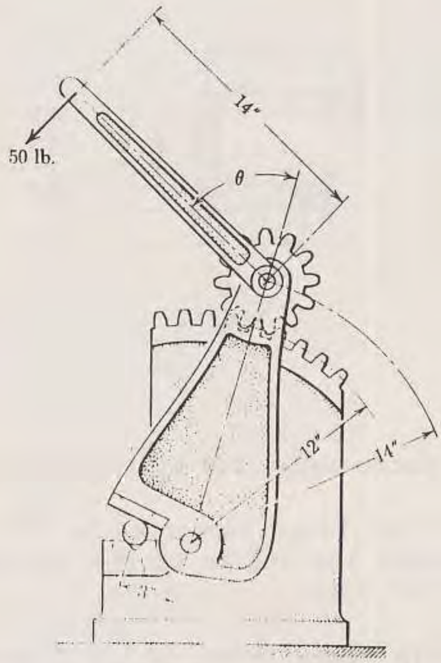


Fig.5

DISTRIBUTED FORCES

25. **Center of Gravity:** The most common distributed force is the force of attraction of the earth. This body force is distributed over all parts of every object in the earth's field of influence. The resultant of this distribution of body force is known as the "WEIGHT" of the body, and it is necessary to determine its magnitude and position for bodies whose weights are appreciable.

Consider a three-dimensional body of any size, shape, and weight. If it is suspended, as shown in Fig.39, by a cord from any point such as A, the body will be in equilibrium under the action of the tension in the cord and the resultant of the gravity or body forces acting on all the particles. This resultant is clearly

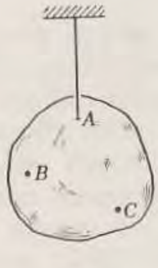


Fig. 39

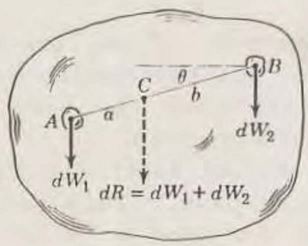


Fig. 40

collinear with the cord, and it will be assumed that its position will be marked, say, by drilling a hole of negligible size along its line of action. The experiment is repeated by suspension at other points such as B and C, and in each case the line of action of the resultant is marked. For all practical purposes these lines of action will be concurrent at a point which is known as the "CENTER OF GRAVITY OR CENTER OF MASS" of the body. An exact analysis, however, would take into account the fact that the directions of the gravity forces for the various particles of the body differ slightly because of the fact that they converge toward the center of attraction of the earth. Also, since the particles are at different distances from the earth, the intensity of the earth's force field is not exactly constant over the body. These considerations lead to the conclusion that the lines of action of the gravity force resultants in the experiments just described will not quite be concurrent, and therefore no unique center of gravity exists in the exact sense. This condition is of no practical importance as long as we deal with bodies whose dimensions are small compared with those of the earth. We therefore assume a uniform and parallel field of force due to the earth's gravitational attraction, and this condition results in the concept of a unique center of gravity.

The position of the center of gravity as a unique point may be determined by experiment, but its existence must not be inferred from this result alone. It may be proved in the following way. Consider any two particles of weight dW_1 and dW_2 in a body as shown in Fig.40. The line AB joining these particles makes an angle θ with the horizontal plane, and the particles are separated by a distance (a+b). The resultant dR of dW_1 and dW_2 is their sum and can be located by Varignon's theorem. Taking moments about B gives:

$$(dW_1 + dW_2)\, b.\cos\theta = dW_1\,(a+b)\cos\theta,$$

or,
$$b = \frac{dW_1}{(dW_1 + dW_2)}\cdot(a+b)$$

The point C on line AB through which the resultant passes is now located, and the result is independent of θ. Hence, regardless of the orientation of the body the resultant of dW_1 and dW_2 will always pass through point C. Combining dR with the weight of a third particle will define another unique point through which the new resultant will always pass. This process is continued until all particles are accounted for and a point is obtained whose coordinates are independent of the orientation of the body. This point is the center of gravity of the body.

To locate mathematically the center of gravity G of any body, Fig.41, an equation may be written which states, by Varignon's theorem, that the moment about any axis of the resultant W of the gravity forces equals the sum of the moments about the same axis due to the gravity forces dW acting on each particle. If the center of gravity is designated by G, its coordinates by x, y, and z, and the total weight by W, the moment principle gives:

$$\left.\begin{aligned} \bar{x} &= \frac{\int x.dW}{W} \\ \bar{y} = \frac{\int y.dW}{W} \quad &; \text{ and } \bar{z} = \frac{\int z.dW}{W} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (17)$$

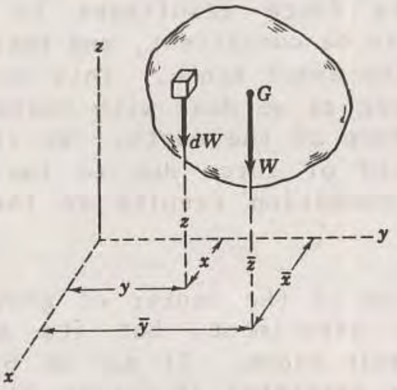


Fig.41

The numerator of each expression represents the sum of the moments, and the product of W and the corresponding coordinate of G represents the moment of the sum. The 3rd equation is obtained by revolving the body and reference frame 90 deg. about a horizontal axis so that the z-axis is horizontal.

26. **Centroids of Lines, Areas, and Volumes:** Whenever the density μ of a body is uniform throughout, it will be a constant factor in both the numerators and denominators of Eqs.17 and will therefore cancel. The expressions then define a purely geometrical property of the body, since any reference to its physical properties is absent. The term "CENTROID" is used when the calculation concerns a geometrical shape only. When speaking of an actual physical body, the term "CENTER OF GRAVITY" or "CENTER OF MASS" is used. If the density is uniform, the positions of the centroid and center of gravity are identical; whereas if the density varies, these two points, in general, will not coincide.

In the case of a slender rod or wire of length L and weight μ per unit length the body approaches a line, and dW = μ.dL. Therefore if μ is constant Eqs.17 become:

$$\bar{x} = \frac{\int x.dL}{L}, \quad \bar{y} = \frac{\int y.dL}{L}, \quad \bar{z} = \frac{\int z.dL}{L} \quad \ldots\ldots\ldots\ldots. (18)$$

When the body has a small thickness and approaches a surface of area A whose weight per unit area is μ, then dW = μdA, and again if μ constant, Eqs.17 give:

$$\bar{x} = \frac{\int x.dA}{A}, \quad \bar{y} = \frac{\int y.dA}{A}, \quad \bar{z} = \frac{\int z.dA}{A} \quad \ldots\ldots\ldots\ldots. (19)$$

If the surface is curved, as with a shell, all three coordinates will be involved. If it is a plane surface, only the two coordinates of that plane will be involved.

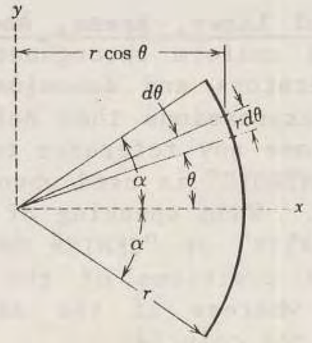
In the case of a body of volume V and density μ the element is dW = μ.dV. The density μ will cancel if it is constant, and:

$$\bar{x} = \frac{\int x.dV}{V}, \quad \bar{y} = \frac{\int y.dV}{V}, \quad \bar{z} = \frac{\int z.dV}{V} \quad \ldots\ldots\ldots\ldots. (20)$$

SAMPLE PROBLEMS

1. Centroid of Circular Arc: Locate the centroid of a line segment in the form of a circular arc as shown in the figure.

Solution: Choosing the x-axis as an axis of symmetry makes y = 0. A differential element of arc has a length dL = $r.d\theta$, and the x-coordinate of the element is $r \cos \theta$.



Applying the first of Eqs.18 gives:

$$[L\bar{x} = \int x.dL]: \quad (2\alpha r)\bar{x} = \int_{-\alpha}^{\alpha} (r \cos \theta) \; .d\theta.$$

$$2\alpha r\bar{x} = 2r^2 \sin \alpha$$

$$\bar{x} = \frac{r \sin \alpha}{\alpha} \qquad \underline{\text{Ans.}}$$

[For $2\alpha = \frac{\pi}{2}$, $\bar{x} = 0.900r$; for a semicircular area $2\alpha = \pi$, which gives $\bar{x} = 2r/\pi$].

---

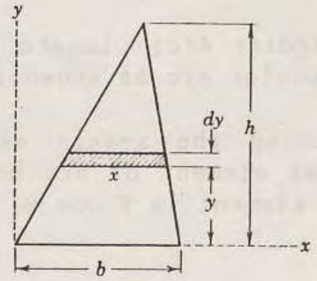
2. Centroid of a Triangular Area: Locate the centroid of the area of a triangle of base b and altitude h.

Solution: The x-axis is taken to coincide with the base. A differential strip of area x.dy is chosen. By similar triangles $\frac{x}{(h-y)} = \frac{b}{h}$. Applying the 2nd Eqs.19 gives:

$$[A\bar{y} = \int y.dA]: \quad \frac{bh}{2}.\bar{y} = \int_0^h y\frac{b(h-y)}{h}.dy$$

$$= \frac{bh^2}{6}$$

$$\bar{y} = \frac{h}{3} \qquad \underline{\text{Ans.}}$$



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3. Centroid of the Area of a Circular Sector: Locate the centroid of the area of a circular sector with respect to its vertex.

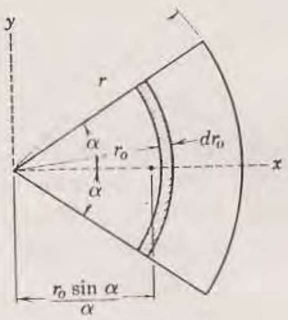
Solution I: The x-axis is chosen as an axis of symmetry, and $\bar{y}$ is therefore automatically zero. The area may be covered by moving an element in the form of a segment of the circular ring, shown in the figure, from the center to the outer periphery. The radius of the ring is r_o and its thickness is dr_o. In the 1st of Eqs.19 the coordinate x is the coordinate to the centroid of the element dA. This distance is $\frac{r_o \sin \alpha}{\alpha}$ (from provious problem no.1) where r_o replaces r. Thus the 1st of Eqs.19 gives:

$$[A\bar{x} = \int x.dA] : \quad \frac{2\alpha}{2\pi}(\pi r^2)\bar{x} = \int_0^r \left(\frac{r_o \sin \alpha}{\alpha}\right)(2r_o.\alpha.dr_o)$$

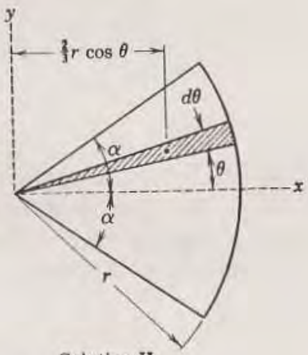
$$r^2.\alpha.\bar{x} = \frac{2}{3}r^3 \sin \alpha$$

$$\bar{x} = \frac{2}{3}\frac{r \sin \alpha}{\alpha} \qquad \underline{\text{Ans}}.$$

[For a semicircular area $2\alpha = \pi$ and $\bar{x} = \frac{4r}{3\pi}$]



Solution I



Solution II

Solution II: The area may also be covered by swinging a triangle of differential area about the vertex and through the total angle of the sector. This triangle, shown in the illustration, has an area $dA = (\frac{r}{2})\ (r.d\theta)$ where higher order terms are neglected. Again the x-coordinate of dA is measured to the centroid of the element, and from previous problem No.2 this coordinate is seen to be $\frac{2r}{3}$ multiplied by cos 0. Applying the first of Eqs.19 gives:

$$[A\bar{x} = \int x dA]: \qquad (r^2\alpha) = \int_{-\alpha}^{\alpha} \left(\frac{2}{3}r \cos \theta\right)\left(\frac{1}{2}r^2 d\theta\right),$$

$$r^2\alpha\bar{x} = \frac{2}{3}r^3 \sin \alpha$$

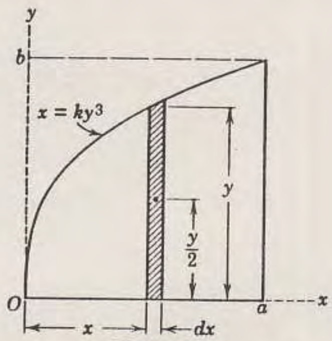
$$\bar{x} = \frac{2}{3}.\frac{r \sin \alpha}{\alpha} \qquad \underline{\text{Ans}}.$$

4. Locate the centroid of the area under the curve $x = ky$ from $x = 0$ to $x = a$.

Solution I: A vertical element of area $dA = ydx$ is chosen as shown in the left part of the figure. The x-coordinate of the centroid is found from the first of equation 19. Thus:

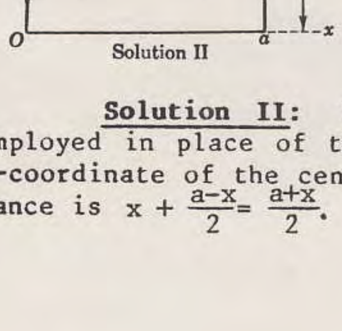
$[A\bar{x} = \int xdA]$:- $\bar{x}\int_0^a ydx = \int_0^a xydx$. Substituting $y = (\frac{x}{k})^{\frac{1}{3}}$ and $k = a/b^3$ and integrating gives:

$$\frac{3ab}{4}\bar{x} = \frac{3a^2b}{7}, \quad \bar{x} = \frac{4}{7}a \qquad \underline{\text{Ans}}.$$



Solution I

In solving for y by the second of Eqs.19 the coordinate y is not the coordinate to the curve $x = ky$ but is the y-distance to the centroid of the element dA. Thus the value , which locates the centroid of the rectangular element, must be used. The moment principle becomes:



Solution II

$[A\bar{y} = \int ydA]$: $\frac{3ab}{4}\bar{y} = \int_0^a (\frac{y}{2})ydx$. Substituting $y = (\frac{x}{k})^{\frac{1}{3}}$ and integrating gives:

$$\frac{3ab}{4}\bar{y} = \frac{3ab^2}{10}, \quad \bar{y} = \frac{2}{5}b \qquad \underline{\text{Ans}}.$$

Solution II: The horizontal element of area shown may be employed in place of the vertical element. In calculating xdA the x-coordinate of the centroid of the element must be used. This distance is $x + \frac{a-x}{2} = \frac{a+x}{2}$. Thus $[A\bar{x} = \int xdA]$:- $\bar{x}\int_0^b (a-x)dy = \int_0^b \frac{a+x}{2})(a-x)\,dy$.

The value of $\bar{y}$ is found from:

$$[A\bar{y} = \int y\,dA] :- \quad \bar{y}\int_0^b (a-x)\,dy = \int_0^a y(a-x)\;dy.$$

The evaluation of these integrals will check the previous results for x and y.

---

5. Hemispherical Volume: Locate the centroid of the volume of a hemisphere of radius r with respect to its base.

Solution: With the axes chosen as shown in the figure $\bar{x} = \bar{z} = 0$ by symmetry. The simplest volume element is a circular slice of thickness by parallel to the x-z plane. Since the hemisphere intersects the y-z plane in the circle $y^2 + z^2 = r^2$ = the radius of the circular slice is $z = +\sqrt{r^2 - y^2}$. The volume of the elemental slice becomes:

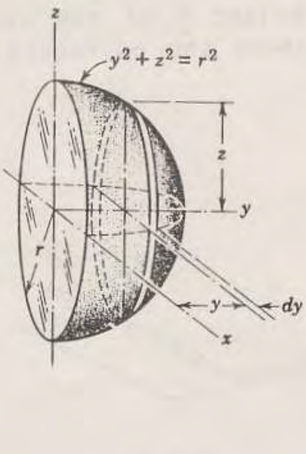
$$dV = \pi\,(r^2 - y^2)\;dy.$$

The second of Eqs.20 requires:

$$[V\bar{y} = \int y\,dV] : \qquad \bar{y}\int_0^r \pi(r^2-y^2)\,dy = \int_0^r y\;(r^2-y^2)\,dy.$$

Integrating gives: $\quad \frac{2}{3}\pi r^3\bar{y} = \frac{1}{4}\pi r^4$

$$\bar{y} = \frac{3}{8}r \qquad \underline{\text{Ans}}.$$



PROBLEMS

1. Locate the centroid of the elliptical area shown.
<u>Ans.</u> $\bar{x} = \frac{4a}{3\pi}$,

$\bar{y} = \frac{4b}{3\pi}$.

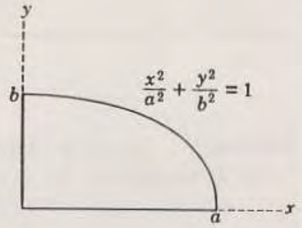


Fig.1

2. Calculate the coordinates of the centroid of the shaded area bound by the circular arc and the chord.

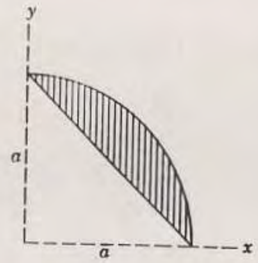


Fig.2

3. Determine the height <u>h</u> of the center of gravity of the cylindrical shell segment above the horizontal surface upon which it rests. <u>Ans.</u> h = 0.45 in.

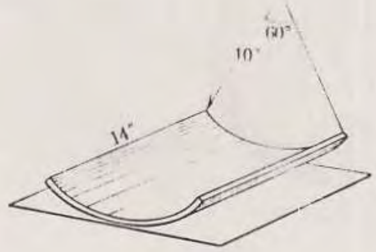


Fig.3

4. Locate the center of gravity of the wire bent into the circular shape shown. Ans. $\bar{x} = -1.273$ in.
$\bar{y} = 1.273$ in.

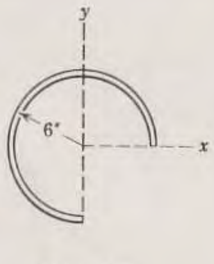


Fig.4

5. A parabola is revolved about the z-axis to obtain the paraboloid shown. Locate the centroid of its volume with respect to the base z = o Ans. $\bar{z} = \frac{a}{3}$

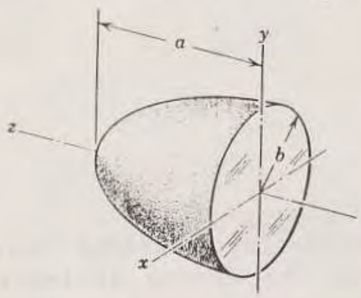


Fig. 5

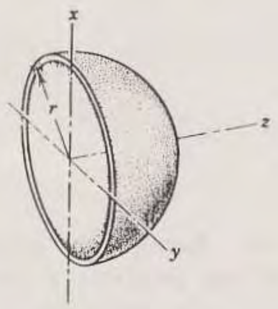


Fig. 6

6. Locate the center of gravity of the homogeneous hemispherical shell of radius r and negligible wall thickness. Ans. $\bar{z} = \frac{r}{2}$

7. Locate the centroid of the area shown in the figure by direct integration. Ans. $\bar{x} = \frac{10-3\pi}{4-\pi} \cdot \frac{a}{3}$

Fig. 7

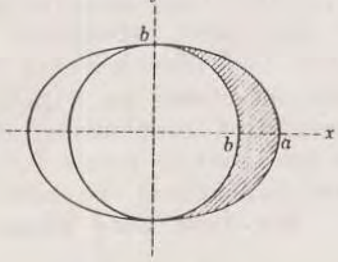


Fig. 8

8. Locate the centroid of the shaded area between the ellipse and the circle. <u>Ans.</u> $\bar{x} = \frac{4(a+b)}{3\pi}$

9. Locate the center of gravity of the bell-shaped shell of uniform but negligible thickness. <u>Ans.</u> $\bar{z} = \frac{a}{\pi-2}$

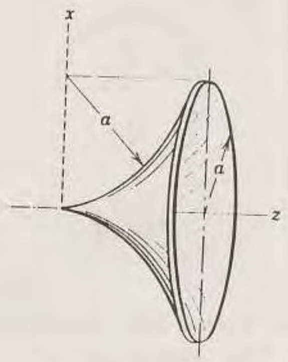


Fig.9

FRICTION

In the foregoing paragraphs, contacting surfaces were considered as being perfectly smooth so that the forces of interaction were normal to the surface. Although in many instances this ideal assumption involves only a very small error, there are a great many problems wherein the ability of contacting surfaces to support tangential as well as normal forces must be considered. Forces tangent to contacting surfaces are known as friction forces and are present to some degree with the interaction between all actual surfaces. Whenever a tendency exists for the sliding of one contacting surface along another surface, the friction forces developed are always found to be in a direction to oppose this tendency.

27. **<u>Dry Friction:</u>** Dry friction is often known as "Coulomb Friction". The laws governing the behaviour of dry friction can be explained for most practical purposes by means of a very simple experiment. Consider a solid block of weight W resting on a horizontal surface as shown in Fig.42a. The contacting surfaces possess a certain amount of roughness. The experiment will involve the application of a horizontal force P. which will vary continuously from zero to a value sufficient to move the block and give it an appreciable velocity. The free body diagram of the block for any value of P is

shown in Fig.42b., and the tangential friction force exerted by the plane on the block is labeled F. This friction force will always be in a direction to oppose motion or the tendency toward motion of the body on which it acts. There is also a normal force N which in this case equals W, and the total force R exerted by the supporting surface on the block is the resultant of N and F. A magnified view of the irregularities of the mating surfaces, Fig.42c., will aid in visualizing the mechanical action of friction. Support is necessarily intermitted and exists at the mating humps. The direction of each of the reactions on the block R_1, R_2, R_3, etc., may be taken to be along the

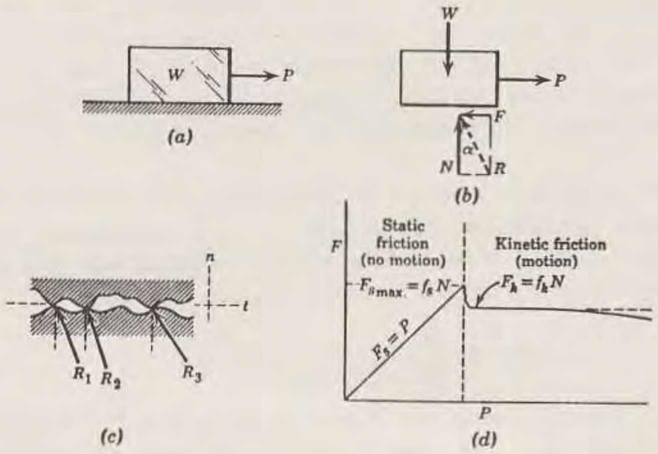


Fig.42

normal to the contact surface of each respective hump. The total normal force N is merely the sum of the n-components of the R's, and the total frictional force F is the sum of the t-components of the R's. When the surfaces are in relative motion, the contacts are more nearly along the tops of the humps, and the t-components of the R's will be smaller than when the surfaces are at rest relative to one another. This consideration will explain the well known fact that the force P necessary to maintain motion is less than that required to start the block when the irregularities are more nearly in mesh.

Assume now that the experiment indicated is performed and the friction force F is measured as a function of P. The resulting experimental relation is indicated in Fig.42d. When P is zero, equilibrium requires that there be no friction force. As P is increased, the friction force must be of equal magnitude to P as long as the block does not slip. During this period the block is in equilibrium, and all forces acting on the block must satisfy the equilibrium equations. Finally a value of P is reached which causes the block to slip and to move in the direction of the applied force. At this same time the friction force drops slightly and abruptly to a somewhat lesser value. Here it remains essentially constant for a period but then drops off still more with higher velocities.

The region up to the point of slippage or impending motion is known as the range of "Static Friction", and the value of the friction force is determined by the equations of equilibrium. This force may have any value from zero up to and including, in the limit, the maximum value. For a given pair of mating surfaces this maximum value of static friction Fs. max is found to be proportional to the normal force N. Thus:

$$F_{s\,max} = f_s N,$$

where f_s is the proportionality constant, known as the "Coefficient of Static Friction". It must be carefully observed that this equation describes only the limiting or maximum value of the static friction force and not any lesser value. Thus the equation applies only to cases where it is known that motion is impending.

After slippage occurs a condition of "Kinetic Friction" is involved. Kinetic friction force is always somewhat less than the maximum static friction force. The kinetic friction force Fk is also found to be proportional to the normal force. Thus:

$$F_k = f_k N,$$

where f_k is the "Coefficient of Kinetic Friction". It follows that f_k is somewhat less than f_s. As the velocity of the block increases, the kinetic friction coefficient decreases somewhat, and when high velocities are reached, the effect of lubrication by intervening air film may become appreciable. Coefficients of friction depend greatly on the exact condition of the surfaces as well as on the velocity and are subject to a considerable measure of uncertainty.

It is quite customary to write the two friction force equation merely as:

$$F = f.N \qquad (21)$$

There will be an understanding from the problem whether limiting static friction with its corresponding coefficient of static friction or whether kinetic friction with its corresponding kinetic coefficient is implied. It should be emphasized again that many problems involve a static friction force which is less than the maximum value at impending motion, and therefore the friction equation cannot be used.

The direction of the resultant R in figure 42b. measured from the direction of N is specified by $\tan\alpha = F/N$. When the friction force reaches its limiting static value, the angle α reaches a maximum value φ_s. Thus, $\tan\varphi_s = f_s$.

When slipping occurs, the angle α will have a value φk corresponding to the kinetic friction force. In like manner:

$$\tan\varphi k = fk,$$

which is customary to write merely,

$$\tan \varphi = f. \qquad (22)$$

where application to the limiting static case or to the kinetic case is inferred from the problem at hand. The angle φ is known as the "Angle of Static Friction", and the angle φk is called the "Angle of Kinetic Friction. This friction angle φ for each case clearly defines the limiting position of the total reaction R between two contacting surfaces. If motion is impending, R must be one element of a right circular cone of vertex angle $2\varphi s$, as shown in Fig.43. If motion is not impending, R will be within the cone. This cone of vertex angle

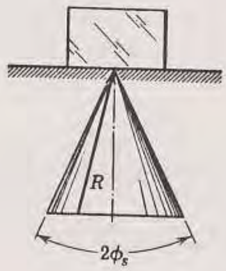


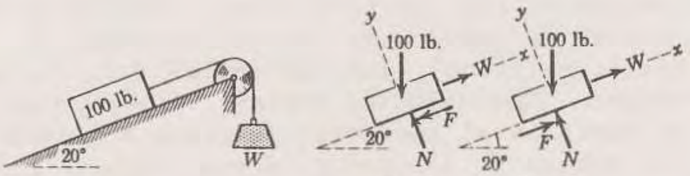
Fig.43

$2\varphi s$ is known as the "Cone of Static Friction" and represents the locus of possible positions for the reaction R at impending motion. If motion occurs the angle of kinetic frictiion applies, and the reaction must lie on the surface of a slightly smaller cone of vertex angle $2\varphi k$. This cone is the "Cone of Kinetic Friction".

Further experiment shows that the friction force is independent of the area of contact. This is true as long as the pressure is not great. For high pressures the surface characteristics are changed and the frictional coefficient increases.

SAMPLE PROBLEMS

1. Determine the range of values which the weight W may have so that the 100 lb. block shown in the figure will neither start moving up the plane nor slip down the plane. The coefficient of static friction for the contact surfaces is 0.30.



Case I Case II

Solution: The maximum value of W will be given by the requirement for motion impending up the plane. The friction force on the block therefore acts down the plane as shown in the free-body diagram of the block for Case I in the figure. Applying the equations of Equilibrium gives:

$[\Sigma Fy = 0]$: $N - 100 \cos 20° = 0, \quad N = 94$ lb.

$[\Sigma F = \int N]$: $F = 0.30 \times 94.0 = 28.2$ lb.

$[\Sigma Fx = 0]$: $W - 28.2 - 100 \sin 20° = 0, \quad W = 62.4$ lb. Ans.

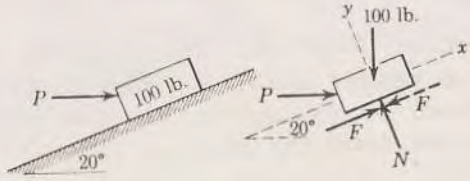
The minimum value of W is determined when motion is impending down the plane. The friction force on the block will act up the plane to oppose the tendency to move as shown in the free-body diagram for Case II. Equilibrium in the x-direction requires:

$[\Sigma Fx = 0]$:- $W + 28.2 - 100 \sin 20° = 0, \quad W = 6.0$ lb. Ans.

Thus W may have any value from 6.0 lb. to 62.4 lb., and the block will remain at rest.

---

2. Determine the amount and direction of the friction force acting on the 100 lb. block shown if, first, P = 50 lb., and second, P = 10 lb. The coefficient of static friction is 0.20, and the coefficient of kinetic friction is 0.17.



<u>Solution:</u> There is no way of telling from the statement of the problem whether the block is or is not on the verge of slipping or whether it is in motion at the position shown. It is therefore necessary to make an assumption, Assume the friction force to be up the plane, as shown by the solid arrow, and the block to be in equilibrium. A balance of forces in both x- and y- directions gives:

$$[\Sigma Fx = 0] : \quad P \cos 20^\circ + F - 100 \sin 20^\circ = 0,$$

$$[\Sigma Fy = 0] : \quad N - P \sin 20^\circ - 100 \cos 20^\circ = 0.$$

<u>Case I</u>: P = 50 lb., substitution into the first of the two equations gives:

$$F = -12.8 \text{ lb.}$$

The negative sign means that, if the block is in equilibrium, the friction force acting on it is in the direction opposite to that assumed and therefore is down the plane as represented by the dotted arrow. Conclusion on the magnitude of F cannot be reached, however, until it is verified that the surfaces are capable of supporting 12.8 lb. of friction force. This may be done by substituting P=50 lb. into the second equation, which gives:

$$N = 111.1 \text{ lb.}$$

The maximum static friction force which the surfaces can support is then:

$$[F = \mathit{f}N] : \quad F = 0.20 \times 111.1 = 22.2 \text{ lb.}$$

Since this force is greater than that required for equilibrium, it follows that the assumption of equilibrium was correct. The answer is then: F = 12.8 lb. down the plane <u>Ans</u>.

<u>Case II</u>: P = 10 lb.

Substitution into the two equilibrium equations gives:

$$F = 24.8 \text{ lb.}; \quad N = 97.4 \text{ lb.}$$

But the maximum possible static friction force is:

$$[F = fN] : \quad F = 0.20 \times 97.4 = 19.5 \text{ lb.}$$

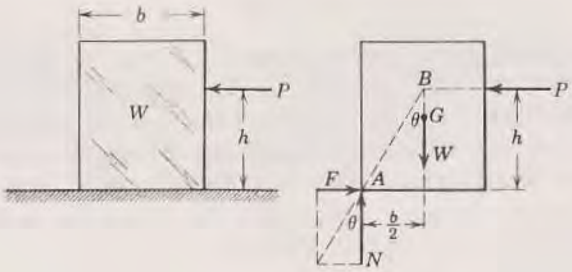
If follows that 24.8 lb. of friction cannot be supported. Therefore equilibrium cannot exist and the correct value of the friction force is obtained by using the kinetic coefficient of friction accompanying the motion down the plane. Thus the answer is:

$$[F = fN] : \quad F = 0.17 \times 97.4 = 16.6 \text{ lb. up the plane} \qquad \underline{\text{Ans}}.$$

It should be noted that, even though ΣFx is no longer equal to zero, equilibrium does exist in the y-direction so that $\Sigma Fy = 0$.

---

3. A homogeneous rectangular block of weight W rests on a horizontal plane and is subjected to the horizontal force P as shown. If the coefficient of friction is f, determine the greatest value which h may have so that the block will slide without tipping.



Solution: If the block is on the verge of tipping, the entire reaction between the plane and the block will be at A. The free-body diagram of the block for this condition is shown in the right side of the figure. If P is sufficient to cause slipping, the friction force is the limiting value fN, and the angle θ becomes $\theta = \tan^{-1} f$. The resultant of F and N passes through a point B through which P must also pass since three coplanar forces in equilibrium are concurrent. Hence from the geometry of the block:

$$\tan \theta = f = \frac{b/2}{h}, \quad h = \frac{b}{2f} \qquad \underline{\text{Ans}}.$$

If h were greater than this value, moment equilibrium about A would not be satisfied. For h less than $\frac{b}{2f}$ the resultant of F and N would be concurrent with P and W at a point below B. Thus this resultant would act not at A but at some point to the right of A.

PROBLEMS

1. Prove whether the block is in equilibrium or whether it is sliding. Find the friction force F acting on the block.
Ans. Slides; F = 2.11 lb.

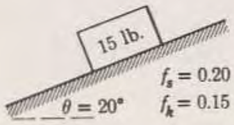


Fig. 1

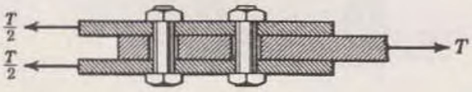


Fig. 2

2. The steel straps shown are held together by two bolts which have been tightened with a torque wrench so that the tension in each bolt is 2000 lb. If the holes in the center strap were drilled slightly oversized, find the tension T which can be supported before the joint slips enough to induce shear in the bolts. Take the coefficient of friction to be 0.2. Ans. T = 1600 lb.

3. Find the total horizontal force S (shear) which the two anchor bolts must exert on the structure to prevent slipping if f = 0.3.

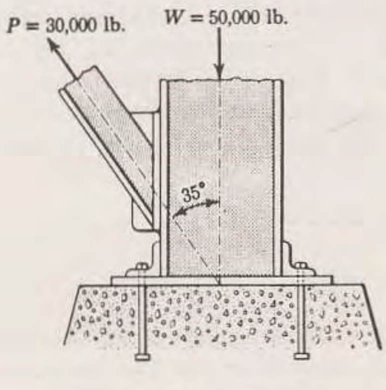


Fig. 3

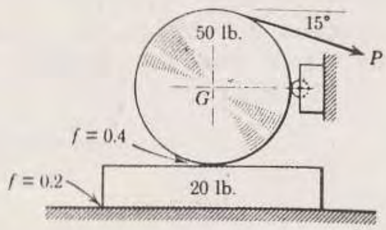


Fig. 4

4. Determine the force P necessary to rotate the 50 lb. cylinder. The coefficient of friction between the cylinder and the 20 lb. block is 0.4 and that between the block and the plane is 0.2
Ans. P = 14.76 lb.

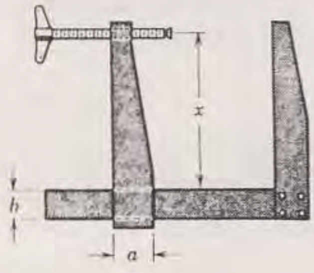


Fig. 5

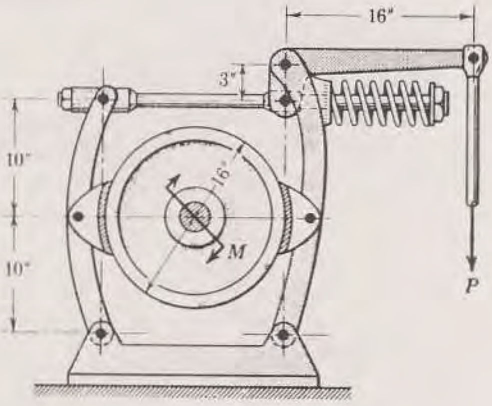


Fig. 6

5. The left-hand jaw of the C-clamp can be slid along the frame to increase the capacity of the clamp. To prevent slipping of the jaw on the frame when the clamp is under load, the dimension x must exceed a certain minimum value. Find this value corresponding to given dimension a and b and a coefficient of friction f between the frame and the loose-fitting jaw.

Ans. $x = \frac{a-bf}{2f}$.

6. The double block brake shown is applied to the flywheel by means of the action of the spring. To release the brake a force P is applied to the control lever. In the operating position with P = 0 the spring is compressed 1 in. Select a spring with an appropriate constant k which will provide sufficient force to brake the flywheel under the torque M = 60 lb.ft. if the coefficient of friction for both brake shoes is f = 0.2. Neglect the dimensions of the brake shoes. Ans. k = 113 lb./in.

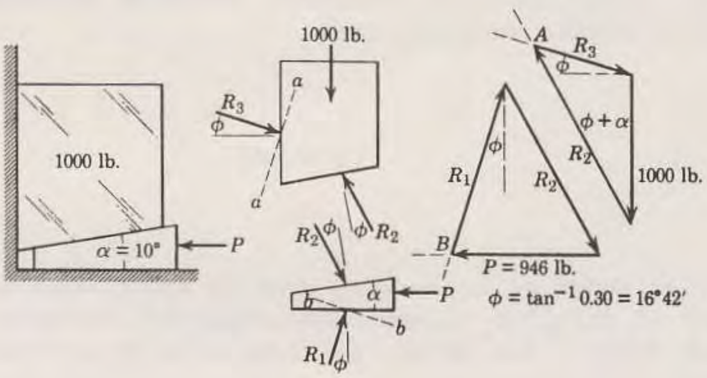
---

28. **Wedges:** A wedge is one of the simplest of machines and is used as a means of producing small adjustments in the position of a body or as a means of applying large forces. Wedges depend upon friction for their action. When the sliding of a wedge is impending, the resultant force on each surface of the wedge will be inclined from the normal to the surface an amount equal to the friction angle. The component of the resultant along the surface is the friction force which is always in the direction to oppose the motion of the wedge.

After drawing the necessary free-body diagram, solution of a wedge problem is usually facilitated by constructing the equilibrium polygon of forces which is normally a triangle. This approach lends itself to a graphical solution, and errors inherent in graphical construction are well within those due to the uncertainty of the friction coefficients. The analysis of wedges is best shown by the solution of a sample problem.

SAMPLE PROBLEM

1. Determine the force P applied to the 10 deg. wedge necessary to start the 1000 lb. block upward. The coefficient of static friction between all surfaces is 0.30, and the weight of the wedge is negligible.



__Graphical Solution:__ The free-body diagrams of the block and wedge are shown in the figure. The reactions R_1, R_2, and R_3 are all inclined with their normals an amount equal to the friction angle φ, since slipping occurs simultaneously at both surfaces, and lean in a direction to oppose the impending motion.

The equilibrium of the block is established by first-laying off the 1000 lb. force to scale, as shown in the upper right part of the figure. Next the known directions of the reactions R_2 and R_3 are used to determine point A, which then establishes the magnitudes of R_2 and R_3. A similar equilibrium triangle for the wedge is constructed by taking the reaction R_2 which is now known and constructing the two lines parallel to the known directions of P and R_1, as shown in the second diagram. The intersection of these two lines determines point B, and thus the magnitude P = 946 lb. is scaled off the drawing.

Algebraic Solution: The simplest choice of axes for the necessary force summations is, in the case of the block, in the direction a-a normal to R_3, and for the wedge in the direction b-b normal to R_1. In this way a simultaneous solution of two equations is avoided. The angle between R_2 and the a-direction amounts to $2\varphi + \alpha = 43°24'$. Thus for the block:

$$[\Sigma Fa = 0] : \quad 1000 \cos (16°42') - R_2 \cos (43°24') = 0,$$
$$R_2 = 1320 \text{ lb.}$$

For the wedge the angle between R_2 and the b-direction is $\frac{\pi}{2} - (2\varphi + \alpha) = 46°36'$. Equilibrium then requires that:

$$[\Sigma Fb = 0] : \quad 1320 \cos (46°36') - P \cos (16°42') = 0,$$
$$P = 946 \text{ lb.} \quad \underline{\text{Ans.}}$$

If the given coefficient of friction between mating surfaces were not all alike, the same procedure would be employed and the modified friction angles would be used.

PROBLEMS

1. A wedge of angle α is used to split wood as shown. If the coefficient of friction between the wedge and the wood is f, determine the maximum angle α for which the wedge will be self-locking.
<u>Ans</u>. $\alpha = 2 \tan^{-1} f$.

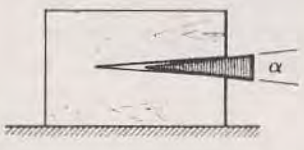


Fig. 1

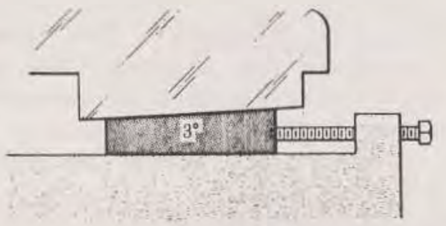


Fig. 2

2. The accurate alignment of a heavy-duty engine on its bed is accomplished by a screw-adjusted wedge with a 3 deg. taper. Determine the horizontal thrust P in the adjusting screw necessary to raise the mounting if the wedge supports 1/4 of the total engine weight of 12000 lb. The coefficient of friction for all surfaces is 0.2.
<u>Ans</u>. P = 1365 lb.

3. In adjusting the position of a vertical column under a load of 5000 lb., two 5 deg. wedges are used as shown. Determine the force P necessary to raise the load if the coefficient of friction of all surfaces is 0.4. Ans. P = 4520 lb.

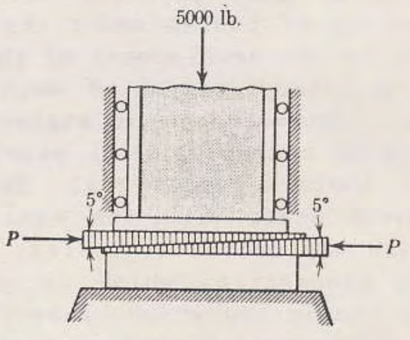


Fig. 3

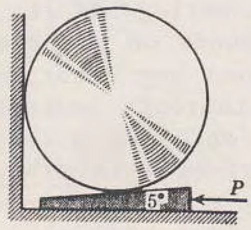


Fig. 4

4. A 5 deg. wedge is used to lift the 1000 lb. cylinder as shown. It the coefficient of friction is 1/4 for all surfaces, determine the force P required to move the wedge.
Ans. P = 378 lb.

DYNAMICS (KINETICS)

Mechanics has been defined as that physical science which deals with the state of rest or motion of bodies under the action of forces. Engineering had its origin in the development of this science and today depends on the continuing interpretation of mechanics probably more than any other subject. The student of engineering will find that a thorough understanding of mechanics will provide one of his most reliable and widely used tools for analysis. Mechanics is logically divided into statics which deals with the equilibrium of bodies and dynamics which deals with the motion of bodies. Dynamics, in turn, has two aspects, first, kinematics, which is a study of motion itself without reference to the forces which cause the motion, and, second, kinetics, which relates the action of forces on bodies to their resulting motions.

Dynamics is a relatively new subject. In terms of engineering application dynamics is an even more recent science. Only since machines have operated with high speeds and appreciable accelerations has it been necessary to make calculations based on the principles of dynamics rather than on the principles of statics. The principles of dynamics are basic to the analysis of moving structures, to fixed structures subject to shock loads, and to practically all types of machinery such as engines, ships, wheeled vehicles, aircraft, rockets, etc. The student whose interests lead him into one or more of these and many other fields will find a constant need for applying his basic knowledge of dynamics.

The vast majority of dynamics problems in engineering may be solved by treating the moving bodies in question as perfectly rigid. The assumption of rigidity means that any relative motions between parts of a body are small compared with the motion of a body as a whole. This book is a study of rigid-body dynamics. The dynamics of non-rigid bodies is a much more difficult subject and one about which a great deal has yet to be learned. The transmission of shock-waves in structures and machines subject to impact loading, for example, is an important problem in nonrigid-body dynamics.

1. **Basic Concepts.** There are certain definitions and concepts which are basic to the study of dynamics, and they should be understood at the outset.

Space is a region extending in all directions. Position in space is determined relative to some reference system by linear and angular measurements. The basic frame of reference for the laws of Newtonian mechanics is the primary inertial system or astronomical frame of reference which is an imaginary set of rectangular axes attached to the mean position of the so-called fixed stars. Measurements show that relative to this reference system the laws of Newtonian mechanics are valid as long as any velocities involved are negligible compared with the speed of light. Measurements made with respect to this reference are said to be absolute, and this reference system is considered to be "fixed" in space. A reference frame attached to the surface of the earth has a somewhat complicated motion in the primary system, and a correction to the basic equations of mechanics must be applied for measurements made relative to the earth's reference frame. In the calculation of trajectories for high-altitude rocket and space flight, for example, the absolute motion of the earth becomes an important parameter. For most engineering problems, however, the corrections are extremely small and may be neglected. On this basis, then, the fundamental laws of mechanics may be applied directly for measurements made relative to the earth, and for most problems the word absolute may be used in a practical sense to refer to such measurements.

Time is a measure of the succession of events and is considered an absolute quantity in Newtonian mechanics. The unit of time is the second, which is a convenient fraction of the period of the earth's rotation.

Force is the action of one body on another. A force tends to move a body in the direction of its action upon it.

Matter is substance which occupies space.

Inertia is the properly of matter causing a resistance to change in motion

Mass is the quantitative measure of inertia.

A body is matter bounded by a closed surface.

A particle is a body of negligible dimensions. In some cases a body of finite size may be treated as a particle, or at other times the particle may be a differential element.

A rigid body is one which exhibits no relative deformation between its parts. This is an ideal hypothesis since all real bodies will change shape to a certain extent when subjected to forces. When such changes are small, the body may be termed rigid without appreciable error.

A scalar quantity is one with which a magnitude only is associated. Examples of scalars are time, volume, density, speed, energy, and mass.

A vector quantity is one with which a direction as well as a magnitude is associated. Examples of vectors are displacement, velocity, acceleration, force, moment, and momentum.

It is assumed at this point that the reader is familiar with the properties of vectors. It is necessary that he understand thoroughly the principle of transmissibility for force vectors, the difference between free, sliding, and fixed vectors, the addition and subtraction of vectors, and the resolution of vectors. A discussion of the properties of vectors may be found in Statics.

2. **Newton's Laws**. Sir Isaac Newton was the first to state correctly the basic laws governing the motion of a particle and to demonstrate their validity. These laws, slightly reworded, are as follows:

Law I. A particle remains at rest or continues to move in a straight line with a uniform velocity if there is no unbalanced force acting on it.

Law II. The acceleration of a particle is proportional to the resultant force acting on it and is in the direction of this force.

Law III. The force of action and reaction between contacting bodies are equal in magnitude, opposite in direction, and collinear.

Newton's second law forms the basis for most of the analysis in mechanics. As applied to a particle of mass $\underline{m}$ it may be stated as:

$$F = ma \qquad (1)$$

where $\underline{F}$ is the resultant force acting on the particle and $\underline{a}$ is the resulting acceleration. This equation is a vector equation since the direction of F must be equal to the direction of $\underline{a}$ in addition to the equality in magnitudes of F and ma. Newton's first law is a consequence of the second since there is no acceleration when the force is zero, and the particle either is at rest or moves with a constant velocity. The first law adds nothing new to the description of motion but is included since it was a part of Newton's classical statements.

The third law is basic to our understanding of force. It states that forces always occur in pairs of equal and opposite forces. Thus the downward force of equal magnitude exerted on the pencil by the desk. This principle holds for all forces, variable or constant, regardless of their source and holds at every instant of time during which the forces are applied.

The mass $\underline{m}$ of a body may be calculated from the results of the simple gravitational experiment. If the gravitational force of attraction or true weight of a body is W, then, since the body will fall with an absolute acceleration $\underline{g}$ in a vacuum, Eq.1 gives:

$$W = mg \quad \text{or} \quad m = \frac{W}{g} \quad \dots\dots\dots\dots\dots\dots \quad (2)$$

3. **Units.** There are a number of systems of units used in relating force, mass, and acceleration. Engineers use a gravitational system in which length, force, and time are considered fundamental quantities and the units of mass are derived. Physicists use an absolute system in which length, mass, and time are considered fundamental and the units of force are derived. Either system, of course, may be used with the same results. The engineer prefers to use force as a fundamental quantity because most of his experiments involve direct measurement of force. The British or FPS gravitational system is the one used in this book. The engineer has not adopted a unit for mass which is universally used although slug and less often g-pound are seen occasionally in the book. One slug (or g-pound) is the mass of a body which weighs 32.2 lb. at the earth's surface.

KINETICS

4. **Introduction.** When a body is subjected to a force system which is unbalanced, the body has accelerated motion. Kinetics is a study of the relations between unbalanced force systems and the changes in motion which they produce.

The basic connection between force and acceleration is stated by Newton's second law of motion. The validity of this law is entirely experimental. Any particle is subjected to the action of a single force F_1. The acceleration a_1 of the particle is measured, and the ratio F_1/a_1 will be some number C_1 whose value depends on the units used for measurement of force and acceleration. the experiment is now repeated by subjecting the same particle to a different force F_2 and measuring the corresponding acceleration a_2. Again the ratio F_2/a_2 will produce a number C_2. The experiment is repeated as many times as desired. Two important conclusions may be drawn from the results. First, the ratios of applied force to corresponding acceleration will all equal the same number, provided the units for measurement are not changed in the experiments. Thus:

$$\frac{F_1}{a_1} = \frac{F_2}{a_2} = \dots\dots\dots = \frac{F}{a} = c, \text{ a constant.}$$

The constant C is a measure of some property of the particle which does not change. This property is the inertia of the particle which is the resistance to change in velocity. Thus for a particle with high inertia (large C) the acceleration will be small for a given force F, and, conversely, if the inertia is small, the acceleration will be large. The mass m is used as the quantitative measure of inertia, and therefore the expression

$$C = km$$

may be written, where k is a constant to account for the units used. Thus the experimental relation becomes

$$F = k\,m\,a \qquad (3)$$

where F is the resultant force acting on a particle of mass m, and a is the resulting acceleration of the particle.

The second conclusion from the ideal experiments is that the acceleration is always in the direction of the applied force. Thus Eq.3 is a vector equation which expresses equality of the direction as well as the magnitude.

It is customary to take k equal to unity in Eq.3 which puts the relation in the usual form of Newton's second law

$$F = m\,a \qquad (4)$$

5. **Motion of a Particle.** Consider a particle of mass m subjected to the action of the concurrent forces F_1, F_2, F_3, ... whose vector sum is ΣF. Equation(4) becomes

$$\Sigma F = m\,a \qquad (5)$$

which is a vector equation with the components

$$\Sigma Fx = m\,a_x$$

$$\Sigma Fy = m\,a_y \qquad (6)$$

$$\Sigma Fz = m\,a_z$$

where $\Sigma F = \sqrt{(\Sigma Fx)^2 + (\Sigma Fy)^2 + (\Sigma Fz)^2}$ and $a = \sqrt{ax^2 + ay^2 + az^2}$. Equation(5) is said to be the equation of Motion for the particle, and Eq.(6) are the three equivalent scalar equations of motion. In applying Eq.(6) the reference axes may be oriented in any convenient manner. Thus, if the x-axis is chosen to coincide with the direction of the resultant acceleration a, Eq.(6) becomes $\Sigma Fx = ma$, $\Sigma Fy = 0$, $\Sigma Fz = 0$. The particle may then be said to be in equilibrium in so far as motion in the y- and z- directions is concerned. The equation of motion may also be written as a differential equation. Thus for motion in the x-direction, for example, the first of Eqs.(6) becomes

$$\Sigma Fx = m \frac{d^2x}{dt^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

This form of the motion equation may be used to describe problems where ΣFx is a function of time and displacement. Two successive integrations of the differential equation produce the relation between x and t. In particular when ΣFx is a function of both x and t, solution of the equation of motion as a differential equation is indicated.

A particle may move on a free path in space without constraints, or it may be constrained to move on a plane or along a line. If free to move in space, such as the center of mass of a rocket in free flight, the particle is said to have "Three degrees of freedom" which means that three independent coordinates are needed to specify the position of the particle at any instant. All three of the scalar equations of motion, Eqs.(6), would have to be applied and integrated to obtain the space coordinates in terms of the time. If a particle is constrained to move in two dimensions, such as a marble sliding on the curved surface of a bowl, only two coordinates are needed to specify its position, and in this case it is said to have "Two degrees of freedom". If a particle in constrained to move along a fixed linear path, such as a bead sliding along a fixed wire, its position may be specified by the coordinate measured along the wire. Thus the particle would have only "One degree of freedom".

Equation(5) gives the instantaneous value of the acceleration corresponding to the forces which act at the moment considered. If the forces are variable, the acceleration will also be variable, and the changes in velocity and displacement of the particle during an interval of its motion may be computed by integrating the variable acceleration in the appropriate kinematical relations $a = dv/dt$ or $a = vdv/ds$.

I. PARTICLE MOTION

6. **Rectilinear Motion.** For rectilinear motion in, say, the x-direction the equation of motion for a particle

$$\Sigma Fx = ma_x$$

is used to determine the acceleration which results from the action of prescribed forces or to find the forces which accompany an acceleration due to a prescribed or constrained motion. If the forces are constant, the acceleration is constant, and vice versa. If the velocity and displacement are desired, the kinematical relations may be used, or else the equation of motion written as a differential equation

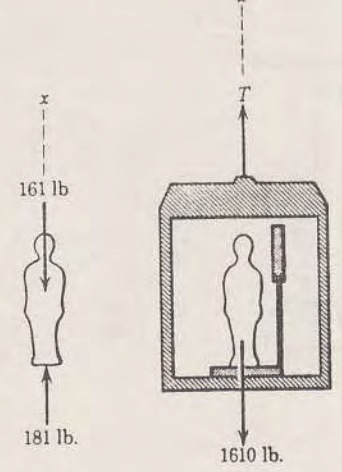
$$\Sigma Fx = m \frac{d^2x}{dt^2}$$

may be integrated successively to find the velocity and the displacement. If the forces are variable, it will be necessary either to integrate Eq.(7) directly or else to solve Eq.(6) for the acceleration as a variable and then to integrate it in one of the kinematical equations $a = dv/dt$ or $vdv = a\,dx$ whichever one is appropriate. The two approaches actually amount to the same process.

If two or more particles or bodies are connected and have the same acceleration, they may be considered together as a single system which eliminates consideration of the internal forces in the connections between them. On the other hand, if these same forces are to be found, the bodies must be considered separately in order to disclose these forces as external loads.

SAMPLE PROBLEMS

1. A 161 lb. man stands on a spring scale in an elevator. During the first 3 sec., staring from rest, the scale reads 181 lb. Find the velocity of the elevator at the end of the 3 sec. and the tension T in the supporting cable for the elevator during the acceleration period. The total weight of elevator, man, and scale is 1610 lb.



Solution. The cable tension and the velocity acquired will depend on the acceleration which is constant since the forces are constant during 3 sec. interval. The acceleration is obtained by considering the two forces acting on the man during the first 3 sec. The free-body diagram of the man is first drawn as indicated. Next the equation of motion is applied which is:

$$[\ Fx = ma\ ] \quad 181 - 161 = \frac{161}{32.2}\,ax; \; ax = 4 \text{ ft/sec.}^2 \text{ up}$$

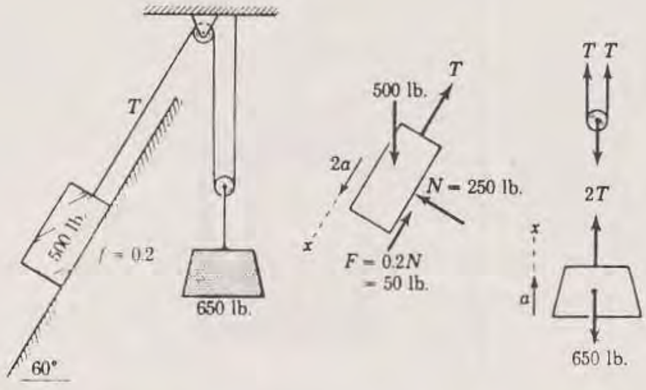
The velocity reached at the end of the 3 sec. is:

$$[v = at] \quad v = 4 \times 3 = 12 \text{ ft/sec.} \qquad \underline{\text{Ans}}.$$

The tension in the cable is obtained from the free-body diagram of the elevator and its contents considered together. Thus:

$$[\ Fx = ma\ ] \quad T - 1610 = \frac{1610}{32.2} \times 4; \; T = 1810 \text{ lb.} \qquad \underline{\text{Ans}}.$$

2. Find the vertical distance S through which the 650 lb. weight has moved during 4 sec. following its release from rest and determine the tension T in the cable. The friction and weight of the pulleys are negligible.



Solution. The distance moved in the given time will depend on the acceleration, which is determined from the force analysis of each of the weights. The direction of movement of the 500 lb. block is not given so must be either assumed or determined. It may be determined by considering the system without friction, and in this event an additional force of 500 sin 60° - (650/2) = 108 lb. would be thus required on the block in the direction up the plane to hold it in equilibrium. This force is greater than the friction force of 0.2 N = 0.2 x 500 x 0.5 = 50 lb., so if the supposed 108 lb. force is released, friction would be insufficient to prevent movement down the plane. Thus the direction of the friction force is established, and the correct free-body diagrams are drawn with the directions of the accelerations indicated. It should be clear that the acceleration of the 500 lb. weight is twice that of the 650 lb. weight. Also with negligible mass of the small pulleys and no appreciable friction in their bearings there is a negligible unbalance of forces and moments required for their accelerations. Hence the small pulleys may be treated as though they were in equilibrium.

The equation of motion for the 500 lb. block is:

$$[\Sigma Fx = ma_x] \qquad 500 \sin 60° - 50 - T = \frac{500}{32.2} \times 2a,$$

and that for the 650 lb. weight is:

$$[\Sigma Fx = ma_x] \qquad 2T - 650 = \frac{650}{32.2}\, a$$

Solution of these two equations gives a = 1.41 ft/sec ., and T = 339 lb. Ans.

Thus in 4 sec. the 650 lb. weight moves:

$$[S = \frac{1}{2}\, at^2] \qquad S = \frac{1}{2} \times 1.41 \times 4^2 = 11.28 \text{ ft. up} \qquad \underline{\text{Ans.}}$$

PROBLEMS

1. Determine the vertical acceleration $\underline{a}$ of the 200 lb. weight for each of the two cases illustrated. The mass and friction of the pulleys are negligible. Ans. (a) $a = 16.1$ ft/sec^2.
(b) $a = 6.44$ ft/sec^2.

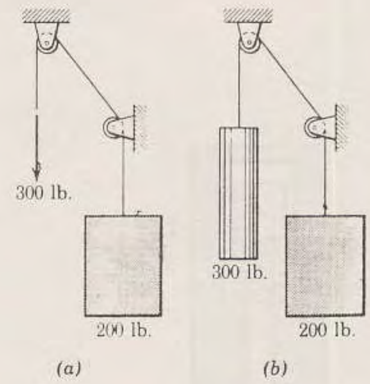


Fig.1

2. The resultant horizontal force on a small object of weight W which moves in a straight line on a horizontal plane is 10 lb. The displacement of the object is given by $S = 4 + 2t + 10t^2$, where S is in feet and t is in seconds. Determine W. Ans. $W = 16.1$ lb.

3. The weight is attached to an arm of negligible mass pivoted to the vertical plate at O. If the plate is given a constant and steady acceleration $\underline{a}$ to the right, determine the angle 0 registered by the pointer. Ans. $0 = \tan^{-1} \frac{a}{g}$

Fig.3

Fig.4

4. The frame is given a steady horizontal acceleration a = 2g. Determine the reaction F between the 10 lb. sphere and the vertical surface. Ans. F = 17.32 lb.

5. A car left skid marks from all four wheels on a level road for a distance of 40 ft. before coming to a stop. Determine the velocity v of the car when the brakes were applied. The coefficient of kinetic friction between the tires and the pavement may be taken as 0.8 .

Fig. 6

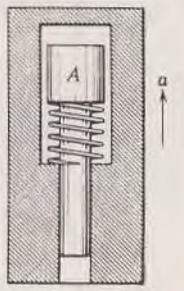


Fig. 7

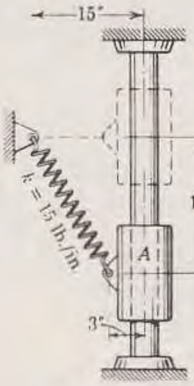


Fig. 8

6. A 150 lb. man hoists himself on a bosun's chair as shown. If, for a short interval, he exerts a pull of 60 lb. on the rope, find his acceleration. Ans. $a = 6.44\ ft/sec^2$. up.

7. The device shown is used as an accelerometer and consists of the 2 oz. plunger A which deflects the spring a measurable amount as the unit is given an upward acceleration a. Specify the necessary spring constant k which will limit the spring compression to 1/4 in., measured from the rest position, for a steady upward acceleration of 4g. Friction is negligible.

8. The collar A weighs 20 lb. and slides on the fixed vertical shaft. The spring is uncompressed when the collar is in the dotted position. Determine the initial acceleration a of the collar when it is released from rest in the position illlustrated. The coefficient of friction between the collar and the shaft is 0.2, and the stiffness of the spring is 15 lb./in. Ans. $a = 99.2\ ft/sec^2$.

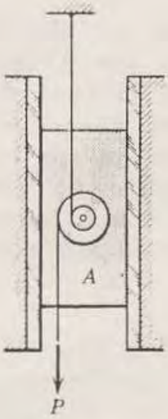


Fig. 9

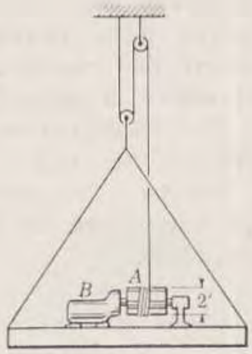


Fig. 10

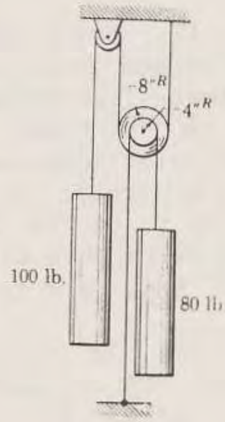


Fig. 11

9. The 32.2 lb. sliding block A moves in the smooth vertical guide under the action of force P applied to the cable as shown. The separate cables are wound around the light pulleys, which are fastened together and which are perfectly free to turn about their common shaft mounted in the block. The diameter of the larger pulley is twice that of the smaller one. Determine the acceleration $\underline{a}$ of the block for (a) P = 35 lb. and (b) P = 30 lb. (Hint : the integral pulley may be treated as a body in equilibrium since its mass is negligible).

Ans. (a) a = 2.8 ft/sec^2. up, (b) a = 2.2 ft/sec^2. down.

10. A platform together with the load it carries weighs 600 lb. and is raised by winding the supporting cable around the drum A, which is driven by the motor and gear unit B. If this unit supplies a starting torque of 250 lb.ft. to the drum, find the initial acceleration $\underline{a}$ of the platform. The weight of the drum is small and may be neglected.

11. Determine the acceleration $\underline{a}$ of the 80 lb. weight for the system shown. The mass of the pulleys and friction are negligible, and the concentric pulleys are free to turn independently of each other about their common axle. Ans. a = 3.58 ft/sec^2.

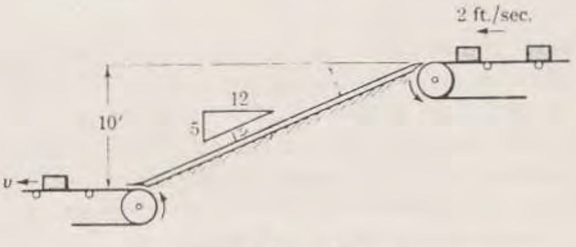


Fig. 12

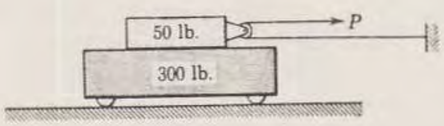


Fig. 13

12. Small objects leave the assembly line at the rate of one every second from a conveyor belt travelling at the linear speed of 2 ft./sec. The objects enter the chute with this initial velocity and slide to the floor below where a second conveyor belt takes them to the shipping department. The coefficient of friction for the parts on the steel chute is found to be 1/3. What would be the required velocity v. of the lower conveyor in order that there be no slipping of the parts when they are deposited horizontally on the belt?

Ans. $v = 11.52$ ft./sec.

13. If the coefficient of friction between the 50 lb. weight and the 300 lb. weight is 0.5, determine the acceleration of each weight for (a) $P = 12$ lb., and (b) $P = 16$ lb. The 300 lb. weight is free to roll, and the weight and friction for the pulley are negligible.

7. **Curvilinear Motion;** n-t components. When a particle moves along a curved path, its components of acceleration both normal and tangent to the path must be accounted for when writing the equation of motion of the particle. The basic vector equation of motion, Eq.(5) is:

$$\Sigma F = ma = m\ (a_n \leftrightarrow a_t)$$

where a is the resultant vector acceleration of the particle and a_n and a_t are its components normal and tangent to the curve, respectively. These components of acceleration were studied in Kinematics and it was found that $a_n = v^2/\rho = \rho w^2 = vw$, and $a_t = \frac{d/v/}{dt} = d|\rho w|/dt = \rho\alpha + w\ d\rho/dt$, where the meaning of each symbol is well known.

The vector equation of motion may be written in terms of its n- and t- components. Thus:

$$\Sigma Fn = ma_n = m\frac{v^2}{\rho}, \qquad \text{............... (8)}$$

$$\Sigma Ft = ma_t = m\frac{d/v/}{dt}$$

The significance of these relations is illustrated in the Fig.1 which shows a particle of mass m moving along any plane curve with increasing speed /v/ in the direction shown. The summation of the n-components of all forces acting on the particle is ΣFn. This sum is always directed toward the center of curvature O since it must have the same direction as a_n which is always toward the center of curvature. The summation of the t-components of all forces acting on the particle is ΣFv. This sum agrees in direction with the tangential acceleration a_t which is the direction of the velocity if the speed is increasing. If the speed is decreasing, a_t and hence ΣFt will be in the direction opposite to the velocity.

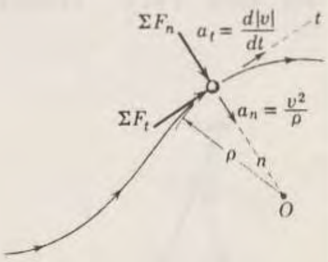


Fig.1

When the motion is circular, the radius of curvature is constant, and the tangential acceleration is merely $a_t = \alpha$. This relation also holds for points on a curved path for which the radius of curvature is a maximum or a minimum given by d /dt = 0.

SAMPLE PROBLEMS

1. Conical Pendulum: A small weight W is suspended by a light arm or wire of length ℓ and made to revolve in a horizontal circle with a constant angular velocity w. Locate the plane of the circular motion by finding h, and calculate the tension T in the supporting member.

Solution. For constant speed of rotation the conical pendulum will assume a position for which θ, h, and r will have fixed values. The free-body diagram of the particle in this position discloses only two real forces acting on it, its weight W and the tension T. The first of Eq.(8) and the equilibrium requirement for the vertical direction give:

$$[\Sigma F_n = ma_n] \qquad T \sin\theta = \frac{W}{g} r w^2,$$

$$[\Sigma F_y = 0] \qquad T \cos\theta = W$$

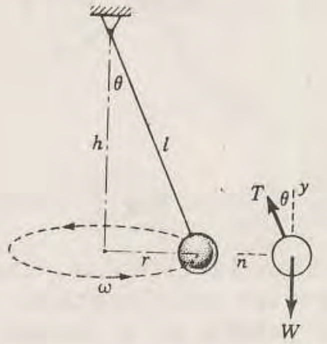
Substitution of $r = \ell \sin\theta$ into the first equation gives:

$$T = \frac{W}{g} \ell w^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \underline{\text{Ans.}}$$

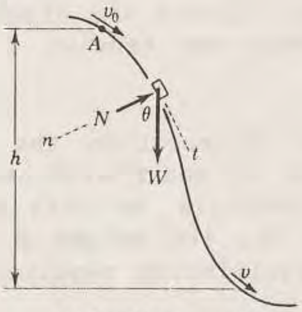
Division of the first equation by the second and substitution of $h = \ell \cos\theta$ give:

$$h = \frac{g}{w^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \underline{\text{Ans.}}$$

This last result shows that the distance h from the plane of rotation to the point of support is the same for all conical pendulums which rotate at the same rate, irrespective of their length ℓ.



The term "Centrifugal Force" is often used (misused) in connection with this and similar problems of rotating bodies. Examination of the free-body diagram of the particle discloses the two forces T and W only. Neither of these forces is centrifugal, which means "away from the center". The word "centripetal" means "toward the center", and therefore the component T sin θ is properly known as a centripetal force. There is no actual centrifugal force acting on the particle.



2. A small object slides on a smooth vertical curve and has an initial velocity vo at a point A as shown. Determine the velocity v. of the object after it has descended a vertical distance h, and write the expression for the normal force acting on the particle at any position.

Solution. The free-body diagram of the particle shows the weight W and the normal force N. The equation of motion in the tangential direction gives:

$$[\Sigma F_t = ma_t] \quad W \sin \theta = \frac{W}{g} a_t, \; a_t = g \sin \theta.$$

The velocity change along the path is given by:

$$[V.dv = a_t.ds] \quad V.dv = g.dS \sin \theta = g.dh.$$

Integration between the appropriate limits gives:

$$V^2 = V^2o + 2gh \; .... \; \underline{\text{Ans}}.$$

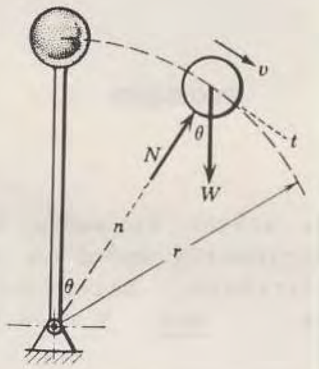
This equation shows that for a path without friction the velocity does not depend on the shape of the path but only on the vertical change in position.

The normal force is given by:

$$[\Sigma F_n = ma_n] \quad W\cos\theta - N = \frac{W}{g}\frac{V^2}{\;} \;;\; N = W\left(\cos\theta - \frac{V^2}{g}\right) \quad \underline{\text{Ans}}.$$

The value of N may be obtained in any particular problem where the radius of curvature ρ is known or can be computed. If friction is present, the problem is complicated considerably.

3. A small sphere of weight W is attached to one end of a light rod freely pivoted about the other end as shown. If the rod and sphere are released from rest in the vertical position, find the angle θ for which the force in the rod is zero, and determine the force in the rod when θ reaches 90 degrees.



Solution: The free-body diagram of the sphere is shown for the general position θ. In addition to the weight W there is the force N which is exerted by the bar on the sphere. This force is along the bar since the bar, if light, may be considered a two-force member. The required answers may be determined when the expression for N as a function of θ is obtained. This expression will depend upon the equations of motion, which are:

$$[\Sigma F_n = ma_n] \quad W\cos\theta - N = \frac{W}{g}\frac{V^2}{r} \;;\; N = W\left(\cos\theta - \frac{V^2}{gr}\right)$$

$$[\Sigma F_t = ma_t] \quad W\sin\theta = \frac{W}{g}a_t; \quad a_t = g\sin\theta.$$

The velocity v of the sphere depends on the tangential acceleration and its change with θ. Thus:

$$[V.dv = a_t.ds) \quad \int_0^v v.dv = \int_0^\theta g \sin\theta \, r.d\theta; \; v^2 = 2gr\,(1-\cos\theta).$$

Substitution of this value for v^2 in the first equation of motion gives:

$$N = (3 \text{ Cos } \theta - 2)\, W$$

The force in the rod is clearly zero when:

$$\theta = \cos^{-1}\frac{2}{3}; \quad 0 = 48° \; 11' \qquad \underline{\text{Ans}}.$$

when $\theta = 90$ deg., $\cos\theta = 0$, and $N = -2W$ <u>Ans</u>.

Hence the force in the rod at this position is a tension equal to twice the weight of the sphere.

PROBLEMS

1. A small 4 lb. weight is swung in a vertical circle of 3 ft. radius with slowly increasing speed on the end of a light steel wire of 100 lb. breaking strength. Determine the velocity V. of the weight when the wire breaks. <u>Ans</u>. V = 48.2 ft/sec.

2. The simple pendulum weighs 4 lb. and is given an initial swing so that its velocity is 10 ft/sec. when θ = 30 deg. Find the tension T in the supporting wire at this instant. <u>Ans</u>. T = 9.67 lb.

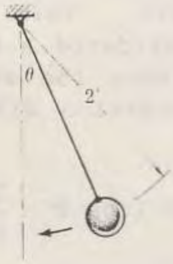


Fig. 2

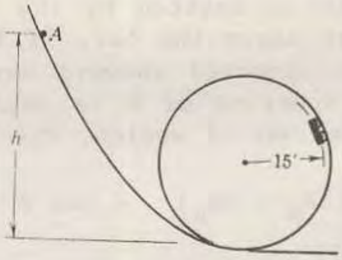


Fig. 3

3. A small car starts from rest at A and rolls freely down the track and around the vertical loop. Determine the minimum height h so that the car does not leave the rails when upside down.
Ans. h = 37.5 ft.

4. Determine the proper angle of bank θ for an airplane flying at 300 mi./hr. and making a horizontal turn of 1 mi radius.
Ans. θ = 48.7 deg.

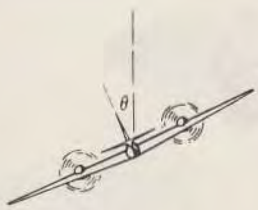


Fig. 4

Fig. 5

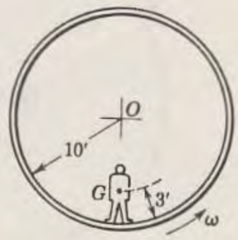


Fig. 6

5. A rocket which moves on a vertical curved path is being propelled by a thrust T of 2000 lb. and is subjected to an atmospheric resistance R of 600 lb. If the rocket has a velocity of 10,000 ft/sec. and if the acceleration of gravity is 20 ft/sec^2. at the altitude of the rocket, find the radius of curvature ρ of its path when in the position shown. Ans. $\rho = 10^7$ ft.

6. The design of a space ship to operate beyond the earth's effective gravitational field is under consideration. If it has the form of a cylindrical shell of 10 ft. radius, determine the angular velocity w of the shell about its central axis O which should be maintained in order to simulate the effect of the earth's gravity for a passenger. Assume the center of gravity of the passenger is 3 ft. from his feet in the standing position shown. Take g to be 32.2 ft/sec^2.

7. At the bottom of a vertical loop the test pilot of an experimental airplane notices that his accelerometer indicates an absolute linear acceleration of the airplane of 5g. normal to its path and that his speed is 600 mi./hr. If the pilot weighs 161 lb., find the radius of curvature ρ of the bottom of the loop and the force N exerted by the man on the seat. Ans. ρ = 4810 ft.; N = 966 lb.

8. The position of the small 1 lb. block in the smooth radial slot of the flywheel depends on the speed of rotation and is used as an activating device for the speed-control mechanism. If the axis of the flywheel is vertical and the block moves from a radius r of 6 in. to one of 7 in. while the speed changes slowly from 300 to 400 rev./min., find the constant k of the spring. Ans. k = 16.46 lb./in.

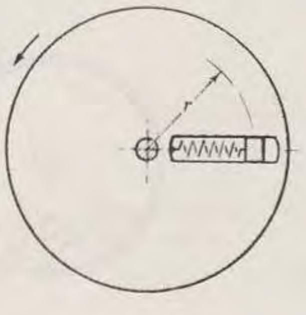


Fig. 8

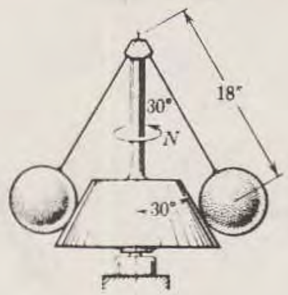


Fig. 9

9. Compute the angular velocity N of the assembly about the vertical axis so that the force of contact R between each 15 lb. sphere and the conical surface is reduced to 4 lb.

II. RIGID-BODY MOTION

8. **Translation.** The translation was defined as any motion in which every line in the body remains parallel to its original position. This requirement means that there can be no angular velocity or angular acceleration of a translating body, and therefore each point in the body has the same acceleration. If a point in such a body moves in a straight line, the body moves with "rectilinear translation". If the point moves along a curve, the body is said to have "curvilinear translation". The principles of kinetics which apply to these two motions are identical.

Figure 2a shows the free-body diagram of a body which has translation in the plane of the figure under the action of external forces F_1, F_2, F_3, ... The mass center G, and hence all points in the body, has a total or absolute acceleration $\bar{a}$. From the principle of motion of the mass center of any system of particles, it is known immediately that the resultant of all external forces is $R = m\bar{a}$, so that two of the equations of motion are $\Sigma Fx = m\bar{a}_x$ and $\Sigma Fy = m\bar{a}_y$ where the x- and y- axes are chosen arbitrarily.

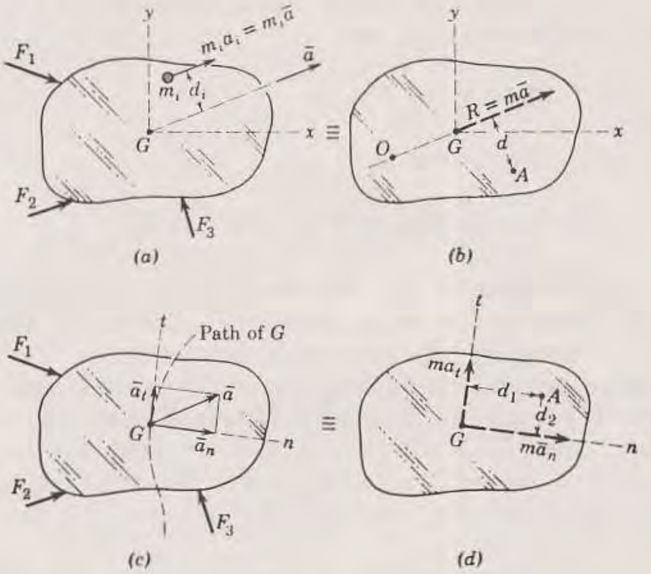


Fig.2

The line of action of R is determined by considering the moment requirement of the forces. The resultant of all forces which act on a representative particle of mass m_i of the body, Fig.2a, is $m_i d_i = m_i a$. The moment of this force about G is m_i a d_i and the sum of the moments about G of all forces acting on all particles is:

$$\Sigma \bar{M} = \Sigma m_i \ \bar{a} \ d_i = \bar{a} \ \Sigma \ m_i d_i$$

By the principle of moments midi = dm where d, the distance to the mass center, is zero. Thus M = 0. In the derivation the moment sum includes the moments of internal and external forces. But since the internal forces always occur in pairs of equal and opposite forces, their net moment is necessarily zero about any point, and M therefore represents merely the sum of the moments of all external forces on the body. From the principle of moments it follows that the moment of the resultant R is also zero, which means that the resultant R = ma must pass through the mass center for a translating rigid body as shown in Fig.2b.

The location of the resultant force through the mass center is the essential characteristic of the forces on a translating body. It may now be observed from Fig.2b that the sum of the moments of all external forces is zero about any point such as O on the line of action of the resultant $R = m\bar{a}$. The moment sum about some point not on this line, such as A, is merely $m\bar{a}d$. This sum is clockwise in the illustration but would be counterclockwise if A were chosen on the opposite side of R. Thus the three equations of motion for a translating rigid body may be written as:

$$\Sigma Fx = m\,\bar{a}_x \qquad\qquad \Sigma Fy = m\,\bar{a}_y \qquad \text{.........................} \quad (9)$$

$$\Sigma\bar{M} = 0 \text{ or } \Sigma Mo = 0 \text{ or } \Sigma MA = m\bar{a}d.$$

It is advisable to represent the resultant $R = m\bar{a}$ either on the free-body diagram or on a separate sketch of the body isolated as in Fig. 2b. Consistent use of a dotted line for the resultant vector will always avoid confusing it with one of the actual applied loads which are represented by full lines. When the resultant is so drawn, the three equations of motion become obvious from the diagram. It is seen that the moment principle may be applied about any point which is convenient, and any desired orientation of axes may be used.

When the translation is curvilinear, the mass center will move on a curve with a total acceleration $\bar{a}$ which for convenience is usually represented in terms of its components in the n- and t- directions, Fig.2c. Thus the motion equations may be written as:

$$\Sigma Fn = m\,\bar{a}_n \qquad \Sigma Ft = m\,\bar{a}_t \qquad \Sigma\bar{M} = 0 \qquad \text{....................} \quad (10)$$

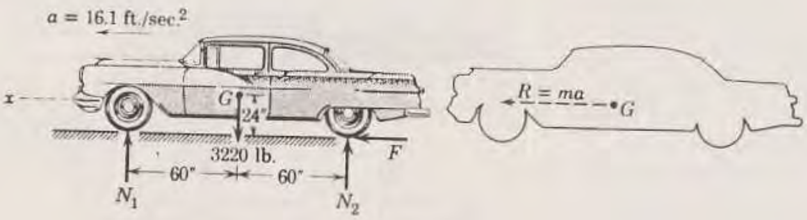
In terms of the velocity v. of the mass center and the radius of curvature ρ of its path, the acceleration components are $\bar{a}n = v^2/\rho$ and $\bar{a}_t = \frac{d|v|}{dt}$.

The resultant force diagram, shown in Fig.2d, may be represented by the n- and t- components of the resultant. It then becomes clear that a moment sum about some convenient point such as A would be $\Sigma M_A = m\bar{a}_t d_1 - m\bar{a}_n d_2$ where the clockwise sense is chosen arbitrarily as positive. The signs in the moment equation will differ depending on the choice of moment center. The student should recognize that the moment equation is nothing more than an application of Varignon's principle which means that the sum of the moments of the external forces shown on the free-body diagram equals the moment of the sum or resultant shown on the resultant-force diagram.

When a translating rigid body is composed of two or more distinct geometrical parts, it is often convenient to replace the resultant $R = m\bar{a}$ by the resultants $R_1 = m_1\bar{a}$, $R_2 = m_2\bar{a}$, ... for each of the separate parts. Each resultant equals the mass of the part times the common acceleration and passes through the mass center of its respective part.

SAMPLE PROBLEMS

1. The 3220 lb. car shown has a forward acceleration on the level road of 16.1 ft/sec . Determine the normal reactions N_1 and N_2 under each pair of wheels, and find the coefficient of friction f between the tires and the road if the rear wheels are on the verge of slipping.



Solution: To free body diagram of the translating car is drawn as indicated. The resultant of the external forces is shown acting through the center of gravity G in the diagram to the right. It will be assumed that the weight of the wheels is small compared with the total weight of the car; otherwise it would be necessary to consider the forces required to produce the angular acceleration of the wheels. The third of the alternate moment relations of Eqs.(9) will eliminate N_2 and F when applied about the rear wheel contact as a moment center. The direction for positive ΣM is counterclockwise about this point since the resultant ma is directed to the left through G. Thus:

$[\Sigma MN_2 = m\ a\ d]$ $\quad 3220 \times 5 - 10\ N_1 = 100 \times 16.1 \times 2,$

$N_1 = 1288$ lb. $\quad$ Ans.

The remaining two principles give:

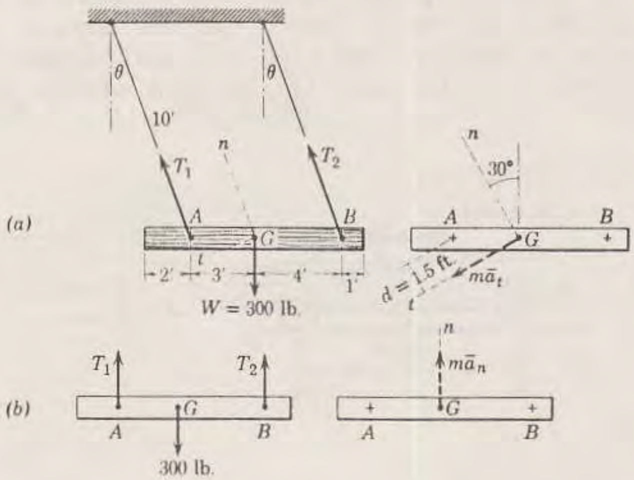
$[\Sigma Fx = ma_x]$ $\quad F = 100 \times 16.1 = 1610$ lb.,

$[\Sigma Fy = 0]$ $\quad 1288 + N_2 - 3220 = 0,\ N_2 = 1932$ Ans.

If the rear wheels are on the verge of slipping, the friction force F is the limiting value, and hence the coefficient of frictiion is:

$$f = \frac{F}{N_2} = \frac{1610}{1932} = 0.834 \qquad \text{Ans.}$$

2. The 300 lb. homogeneous log is supported by the two parallel 10 ft. ropes. It is released from rest in the position for which θ = 30 deg. and swings past the bottom position. Determine the tensions T_1 and T_2 in the ropes (a) an instant after release and (b) as the bottom position is reached.



Solution: The tensions are disclosed by the free-body diagrams of the log for each position. Points A and B, and hence G, have identical curvilinear motion, so that the log moves with curvilinear translation, always parallel to its original position. For condition (a) immediately after release the velocity of G is essentially zero, so that its acceleration has no n-component. The resultant force diagram on the right shows $m\bar{a}_t$ acting through G in the direction of the tangential component of acceleration. For condition (b) the forces are vertical, and the resultant has an n-component only as shown on the resultant force diagram for this case.

The tangential acceleration is obtained from the first free-body diagram. Thus:

$$[\Sigma F_t = m\bar{a}_t] \qquad W \sin\theta = \frac{W}{g}\,\bar{a}_t,\ \bar{a}_t = g \sin\theta.$$

which for θ = 30 deg. is $\bar{a}_t = 0.5\ g\ \text{ft/sec}^2$. A moment equation about A will yield T_2, or one about B will yield T_1. With A as a moment center the resultant force diagram shows that the moment sum is clockwise. Thus for the 30 deg. position:

$$[\Sigma M_A = m\bar{a}_t d] \quad 300 \times 3 - 7\ T_2 \cos 30° = \frac{300}{g} \times 0.5\ g \times 1.5,$$

$$T_2 = 111\ \text{lb.} \quad \underline{\text{Ans.}}$$

The tension T_1 may be found by a similar equation about B or by a force summation in the n-direction. The latter step gives:

$$[\Sigma Fn = 0] \quad T_1 + 111 = 300 \cos 30°, \quad T_1 = 149 \text{ lb.} \qquad \underline{\text{Ans.}}$$

The forces acting at condition (b) depend upon the normal acceleration which, in turn, depends upon the velocity. The velocity is obtained by integrating the tangential acceleration. Thus $[v.dv = a_t ds]$ $\int_0^v v.dv = \int_{30°}^{0} g \sin\theta\,(-10\, d\theta)$; $v^2 = 2.68\, g\, (\frac{bt}{sec})^2$. and $\bar{a}_n = v^2/\rho = 2.68\, g/10 = 0.268\, g$ ft./sec.^{2}. A moment sum about B is clockwise as seen from the resultant force diagram. Hence,

$$[\Sigma M_B = m\bar{a}_n d] \quad 7\,T_1 - 300 \times 4 = \frac{300}{g} \times 0.268\, g \times 4, \; T_1 = 217 \text{ lb.} \qquad \underline{\text{Ans.}}$$

The tension T is obtained from:

$$[\Sigma F_n = ma_n] \quad 217 + T_2 - 300 = \frac{300}{g} \times 0.268\, g, \; T_2 = 163 \text{ lb.} \qquad \underline{\text{Ans.}}$$

A moment equation about A will yield the same result. It should be noted that a moment sum given by $\Sigma\bar{M} = 0$ could have been applied for either position but would have necessitated a simultaneous solution with the force equation since each relation would involve both unknown tensions. Other choices of moment centers may be noted. For instance, in case (a) a zero moment summation would result about the point of intersection of T_1 and the resultant $m\bar{a}_t$ or about the intersection of T_2 and $m\bar{a}_t$. Either of these two points would represent a good choice of moment center and would permit the direct calculation of a tension without involving the acceleration.

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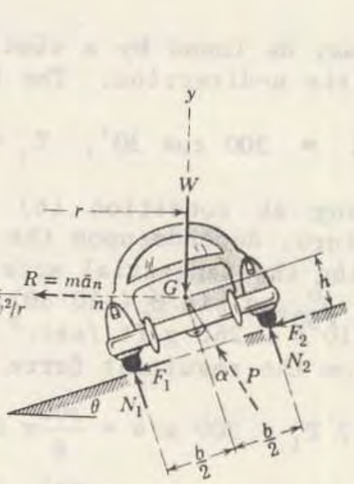
3. Investigate the relations between the angle to which a curved road is banked and the tendency for a car rounding the curve to tip over or slide.

Solution: The rear view of a car rounding an inwardly banked curve of mean radius *r* at a constant speed v is shown. The velocity of the car is normal to the plane of the figure, but the acceleration, $an = v^2/r$, is toward the center of the curve and is in the plane of the paper. If *r* is large compared with the dimensions of the car, each point in the car may be assumed to have the same acceleration. Thus the car may be analyzed by the principles of translation applied in the plane of the figure even though the actual motion is normal to this plane. The forces acting on the car may be represented by the



weight W and the force P which is the resultant of the normal forces N_1 and N_2 and the lateral friction forces F_1 and F_2. Each of these wheel forces is, of course, the sum of the front and rear-wheel forces. The force P must pass through G since the resultant of P and W is $R = m\bar{a}_n$ which passes through G. The equations of motion are:

$$[\Sigma Fn = m\bar{a}_n] \qquad P \sin(\theta + \alpha) = \frac{W}{g}\frac{v^2}{r},$$

$$[\Sigma Fy = 0] \qquad P \cos(\theta + \alpha) = W$$

Dividing gives: $\tan(\theta + \alpha) = \frac{v^2}{gr}$ or $v^2 = gr\,\frac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha}$

The angle of bank which produces no tendency to tip or slip for a particular speed v is that angle for which there is no side friction. Thus $\alpha = 0$, $N_1 = N_2$, and $\tan\theta = \frac{v^2}{gr}$.

This relation shows that a road can be properly banked for one speed only. The speed at which the car overturns occurs when the reaction P acts entirely at the outside wheels. In this event $\tan\alpha = \frac{b}{2}/h$, and thus:

$$v^2 = gr\,\frac{\tan\theta + (\frac{b}{2h})}{1 - (b/2h)\tan\theta}$$

This relation assumes sufficient friction to allow P to act at the outer wheels and is valid provided the coefficient of friction f is greater than $\frac{b}{2h}$.

The car will slide before it will tip, on the other hand, if the coefficient of friction f is less than $(b/2h)$. Thus $\tan\alpha = f$, and the speed at which sliding begins is given by:

$$v^2 = gr\,\frac{\tan\theta + f}{1 - \quad \tan\theta}$$

PROBLEMS

1. The device shown consists of a vertical frame A to which are pivoted a geared sector at O and a balanced gear and attached pointer at C. Determine the relation between the steady horizontal acceleration a, expressed as a fraction of g, and the angle θ registered by the pointer. Friction may be neglected.

<u>Ans.</u> $\frac{a}{g} = \tan 0.214\,\theta$

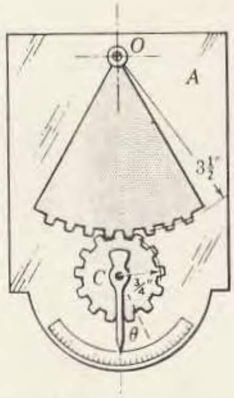


Fig. 1

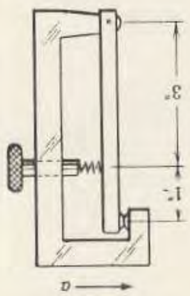


Fig. 2

2. One type of instrument for measuring accelerations is known as a classifying accelerometer and indicates whether an acceleration is greater or less than some prescribed value. The device shown may be used as such an instrument. When the acceleration to the left exceeds a certain critical value, the uniform bar of weight W rotates slightly against the spring and opens the electrical contacts. If the bar weighs 8 oz. and the spring has a stiffness of 20 lb./in., how many turns N of the adjusting screw from the position of initial contact of the spring with the bar are required to preset the device for an acceleration of 12g? The screw has 40 single threads per inch. <u>Ans.</u> 8 turns.

3. At what speed can a car round a turn of 100 ft. radius on a flat unbanked road without slipping if the coefficient of friction between the tires and the road is 0.8 and if the center of gravity of the car is sufficiently low to prevent overturning?

<u>Ans.</u> v = 34.6 mi/hr.

4. The center of gravity of a certain car is 2 ft. from the road, and the tread (transverse distance between wheels) is 6 ft. Also the coefficient of friction between the tires and the road is 0.80. What is the maximum speed v with which the car can enter a turn of 100 ft. radius banked inward at an angle of 15 deg. without tipping or sliding? Which would occur first? Ans. v = 45.1 m /hr, sliding.

5. The uniform bar AB weighs 100 lb. and is pinned at A and fastened by a cable at B to the frame F. If the frame is given an acceleration a = 0.5g, determine the tension T in the cable and the total force exerted by the pin at A on the bar.

Ans. T = 35.4 lb., A = 79.1 lb.

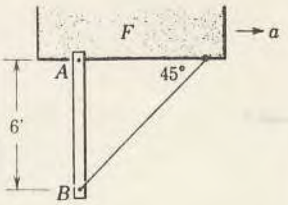


Fig. 5

6. What are the minimum speed v. and corresponding angle θ in order that the motorcycle may ride on the vertical wall of the cylindrical track? The coefficient of friction between the tires and the wall is 0.70. Ans. v = 25.3 mi/hr., θ = 55 deg.

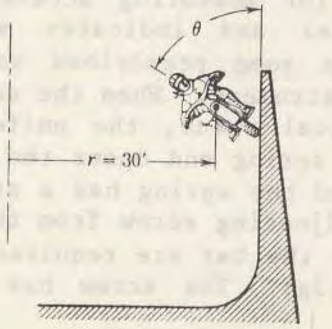


Fig. 6

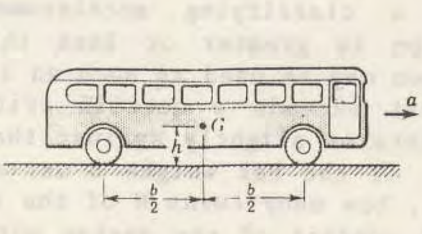


Fig. 7

7. Find the maximum velocity v which the bus can reach in a distance S from rest without slipping its rear driving wheels if the coefficient of friction between the tires and the road is f. Neglect the weight of the wheels.

Ans. $v = \sqrt{\frac{fbg.S}{b-fh}}$.

8. By reversing the pitch of its propellers upon landing, the transport plane reduces its speed from 100 mi./hr. to 30 mi./hr. in 800 ft. of runway length with constant deceleration. Determine the force P on the front wheels at the end of this interval if no mechanical braking forces are applied to the wheels. The plane weighs 150,000 lb. with center of gravity at G. The aerodynamic forces at 30 mi./hr. may be neglected except for the resultant negative propeller thrust which passes through G. Does P depend on the acceleration?

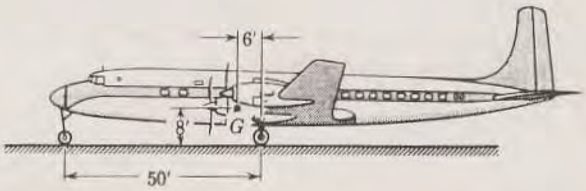


Fig. 8

9. If the 150,000 lb. transport plane shown with problem 8 is brought to a stop from a velocity of 30 mi/hr. in 80 ft. with its wheel brakes, determine the normal force P under the front wheels. The propellers are idling, and other aerodynamic forces are negligible at the low speed involved. <u>Ans</u>. P = 27,000 lb.

10. A water heater is placed on the horizontal bed of a truck. The base of the four legs forms a 12 in. square as shown, and the center of gravity of the heater is 3 ft. above its base. The coefficient of friction between the legs and the truck bed is 0.25. If the driver forgets to secure the heater with the ropes shown, find the forward acceleration a at which heater tips or slips.
<u>Ans</u>. $a = 5.37$ ft/sec^2, tips.

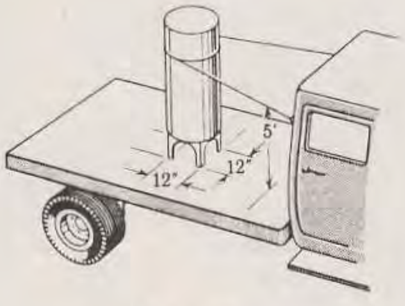


Fig. 10

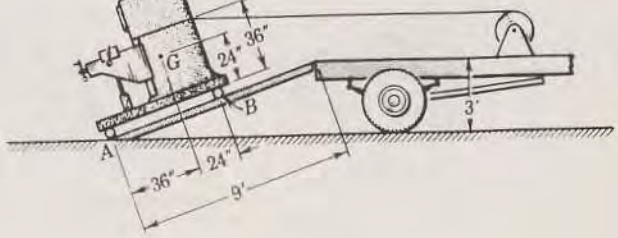


Fig. 11

11. A 3000 lb. milling machine with center of gravity at G is to be hoisted on rollers from the position shown to the bed of the truck. When the power for the winch is applied, the cable tension is momentarily 50 per cent greater than that necessary to maintain equilibrium of the milling machine with both rollers on the incline. Determine the reactions on the rollers A and B at the instant of this maximum tension. Neglect friction on the rollers compared with the other forces acting. Ans. A = 831 lb., B = 2530 lb.

12. A force P is applied to the homogeneous rectangular box of weight W. If the coefficient of friction is f, determining the limiting values of h so that the box will slide without tipping about either the front edge or the rear edge.
Ans. $h = \frac{1}{2}\,[b - \frac{W}{P}\,(fb \pm c)]$

9. **Fixed Axis Rotation:** Consider any rigid body, Fig.3a, which rotates about a fixed axis through point O. The plane of rotation is normal to this axis and passes through the center of mass of the body. At the instant considered the body is assumed to have an angular velocity w and an angular acceleration α. The projections of the external forces acting on the body onto the plane of rotation are indicated by the full arrows on the free-body diagram, Fig.3b., and include the reaction exerted by the bearing on the body at O. If the plane of rotation is other than horizontal, the weight W of the body will appear on the free-body diagram. The acceleration $\bar{a}$ of the mass center G has the components $\bar{a}_n = \bar{r}w^2$ and $\bar{a}_t = \bar{r}\alpha$. Thus from the principle of motion of the mass center, it is known immediately that

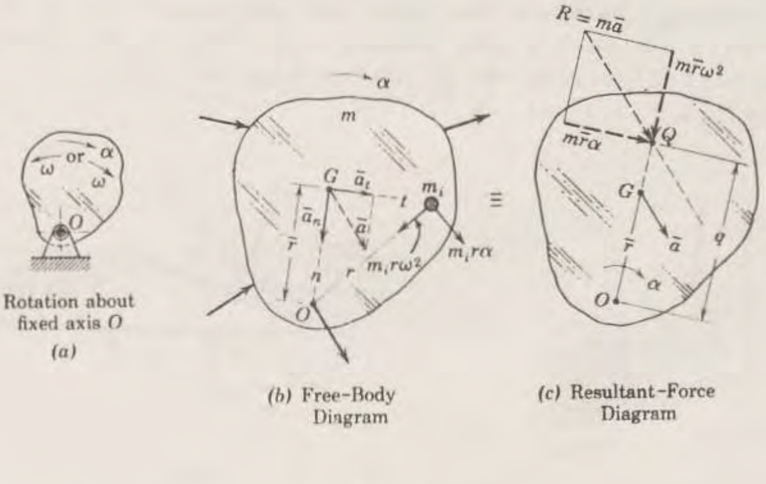


Fig. 3

the resultant R of all external forces acting on the body is $\Sigma F = R = m\bar{a}$ and is in the same direction as $\bar{a}$. The resultant is represented by its two components is Fig.3c, so that $\Sigma Fn = mrw^2$ and $\Sigma F_t = mr\alpha$. It must not be assumed that the resultant passes through G for rotation as it does for translation. The location of the line of action of R is determined by evaluating the sum of the moments of all forces about O.

The moment relation is obtained by considering the forces acting on a representative particle of mass m_i. The resultant of these forces may be represented by its two components $m_i r w^2$ and $m_i r \alpha$ as shown in Fig.3b. Of the two components only the tangential one exerts a moment about O, and the magnitude of this moment is $m_i r^2 \alpha$. If this moment is added to those for the remaining particles in the body, the sum:

$$\Sigma Mo = \Sigma m_i r^2 \alpha = \alpha \Sigma m_i r^2$$

results. The acceleration α is common to all terms and may be factored outside the summation sign. The sum ΣMo includes moments due to internal forces (actions and reactions between particles) and moments due to external forces. Since each internal action is accompanied by an equal and opposite internal reaction, it follows that the net contribution to ΣMo by the internal forces is zero. Therefore the expression represents the algebraic sum of the moments about the axis of rotation of all external forces. The summation $\Sigma m_i r^2$ depends on the radial distribution of mass about the axis and is known as the "mass moment of inertia I" of the body about O.

The three equations of motion for a rigid body rotating about a fixed axis through O may now be written as:

$$\begin{aligned} \Sigma Fn &= m\,\bar{r}\,w^2, \\ \Sigma Ft &= m\,\bar{r}\,\alpha\,, \qquad \ldots\ldots\ldots\ldots\ldots (11) \\ \Sigma Mo &= Io\,\alpha \end{aligned}$$

The use of these equations of motion is straight forward when they are applied exactly and literally with the aid of a complete and correct free-body diagram.

In terms of the differential element of mass dm, the mass density , and the volume element dV the defining expression for mass moment of inertia may be written as:

$$Io = \Sigma m_i r^2 \quad = \quad r^2 dm \quad = \quad r^2 dV,$$

where the integral is evaluated over the entire volume of the body. The radius of gyration k of the body about the axis through O is defined by:

$$k^2 o \; = \; \frac{Io}{m}$$

Moments of inertia are involved in all problems of bodies which have rotational acceleration, and it is necessary to be familiar with them in order to proceed further. A detailed discussion of mass moments of inertia could be found in several text-books.

The line of action of the resultant R for a rotating rigid body, Fig.3c, may be found by locating point Q. The principle of moments requires that the sum of the moments of all external forces on the body must equal the moment of their sum or resultant. Thus:

$$Io\alpha = qm\bar{r}\alpha, \; k^2 om\alpha = qm\bar{r}\alpha$$

so that: $$q = \frac{k^2 o}{\bar{r}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

The point Q, located by the distance q from O, is known as the center of percussion about O. With Q located two alternate moment equations which are often convenient may be written. The first, a moment sum about Q is clearly zero. The second, a moment sum about the mass center G, is obtained by multiplying $\Sigma F_t = m\bar{r}\alpha$ by $q-\bar{r}$ which gives:

$$m\bar{r}\alpha \; (q-\bar{r}) = m\alpha \; (k^2 o - \bar{r}^2) = m\bar{k}^2\alpha = \bar{I}\alpha$$

Thus two alternate moment equations of motion for a rigid body rotating about a fixed point are:

$$\Sigma M_Q = 0 \qquad \text{and } \Sigma\bar{M} = \bar{I}\alpha$$

Points other than O, G, or Q may be used as moment centers if desired, and the correct expression for the moment sum may be determined from the resultant force diagram by using the principle of moments.

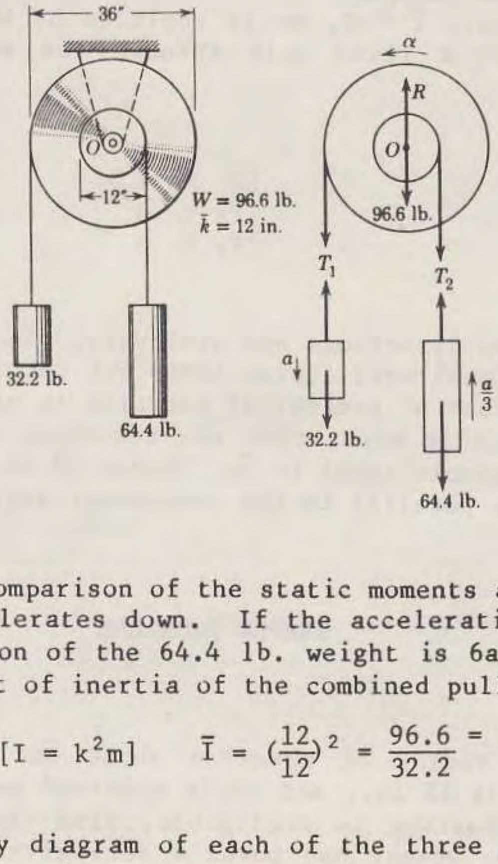
For the special case of rotation of a rigid body about a fixed centroidal axis $\bar{r} = 0$, Mo is replaced by $\bar{M}$ and Io by $\bar{I}$. Thus for rotation about a fixed axis through the mass center Eqs.(11) become:

$$\Sigma M = \bar{I}\,\alpha$$
$$\Sigma Fx = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$
$$\Sigma Fy = 0,$$

where the x- and y- directions are arbitrary. Such a body may be said to be in translational equilibrium ($\bar{a}$=0) but not in rotational equilibrium. For this case of centroidal rotation it may be observed, since the resultant force is zero, that the resultant of all forces acting on the body is a couple equal to $\bar{I}\alpha$. Hence $\Sigma\bar{M}$ is the same as a moment sum about any axis parallel to the centroidal axis.

SAMPLE PROBLEMS

1. The radius of gyration about the axis of the uniform integral pulleys is 12 in., and their combined weight is 96.6 lb. If friction in the bearing is negligible, find the distance S through which the 32.2 lb. weight has moved 4 sec. after released from rest. The cables are wrapped securely around the pulleys. Also find the bearing reaction at O during this interval.



Solution: Comparison of the static moments about 0 shows that the 32.2 weight accelerates down. If the acceleration of this weight is a, the acceleration of the 64.4 lb. weight is 6a/18 upward. Also the centroidal moment of inertia of the combined pulleys is:

$$[I = k^2 m] \qquad \bar{I} = (\frac{12}{12})^2 = \frac{96.6}{32.2} = 3 \text{ lb.ft.sec}^2.$$

The free-body diagram of each of the three members in shown. For the two weights:

$$[\Sigma F = ma] \qquad 32.2 - T_1 = \frac{32.2}{32.2} a; \; T_2 - 64.4 = \frac{64.4}{32.2} \cdot \frac{a}{3}.$$

For centroidal rotation of the pulleys:

$$[\Sigma \bar{M} = \bar{I} \;] \qquad \frac{18}{12} T_1 - \frac{6}{12} T_2 = 3 \frac{a}{18/12}$$

Solution of the three equations gives:

$$a = 4.20 \text{ ft/sec}^2, \quad T_1 = 28.0 \text{ lb.}, \quad T_2 = 67.2 \text{ lb.}$$

With the constant acceleration known the distance dropped by the 32.2 lb. weight in 4 sec. is:

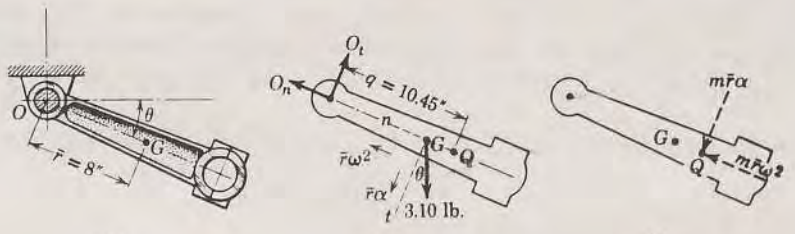
$$[S = \frac{1}{2} at^2] \qquad S = \frac{1}{2} \times 4.20 \times 4^2 = 33.6 \text{ ft.} \qquad \underline{\text{Ans}}.$$

The bearing reaction is obtained from the vertical equilibrium of forces on the drum and is:

$[\Sigma F_y = 0] \quad R - 96.6 - 28.0 - 67.2 = 0; \; R = 191.8 \text{ lb.}$ <u>Ans.</u>

---

2. The center of gravity of the 3.10 lb. connecting rod shown in part (a) of the figure is at G, and the radius of gyration of the rod about the pivot axis 0 is 9.14 in. If the rod is released from rest with 0 = 0, find the total force on the bearing 0 when the position 0 = 45 deg. is passed. Neglect any friction in the bearing.



(a) (b) (c)

Solution: The free body diagram of the rod in an intermediate position 0 is shown in part (b) of the figure, and the resultant of the external forces is represented in part (c) of the figure. The bearing reaction is indicated by its n- and t- components where the sense of these components may be assumed for the present. The normal component O_n is found from a force equation in the n-direction which involves the normal acceleration $\bar{r}\, w^2$. Since w is found from the integral of the angular acceleration and since O_t depends on the tangential acceleration $\bar{r}\alpha$, it follows that α must be obtained first. The moment equation about 0 gives:

$$[\Sigma M_o = I_o\alpha] \quad 3.10 \times \frac{8}{12} \cos\theta = \left(\frac{9.14}{12}\right)^2 . \frac{3.10}{32.2}\,\alpha; \quad \alpha = 3.70 \cos\theta$$

Then $[w\,dw = \alpha\, d0]$:

$$\int_0^w w\, dw = \int_0^{\frac{\pi}{4}} 37.0 \cos\theta . d\theta,$$

$$w^2 = 52.3 \text{ (rad./sec.)}^2$$

The remaining two equations of motion applied to the 45 deg. position yield:

$$[\Sigma Fn = m\bar{r}w^2] \quad O_n - 3.10 \times 0.707 = \frac{3.10}{32.2} \times \frac{2}{3} \times 52.3, \; O_n = 5.55 \text{ lb.}$$

$$[\Sigma F_t = m\bar{r}\alpha] \quad 3.10 \times 0.707 - O_t = \frac{3.10}{32.2} \times \frac{2}{3} \times 37.0 \times 0.707, \; O_t = 0.51 \text{ lb.}$$

The total bearing force is:

$$O = (5.55)^2 + (0.51)^2 = 5.57 \text{ lb.} \quad \underline{\text{Ans}}.$$

The proper sense for O_t may be observed at the outset by applying the alternate moment equation, $\Sigma\bar{M} = \bar{I}\alpha$. This relation eliminates all forces but O_t and requires a clockwise moment to agree with the known direction of α. Also, the component O_t may be found directly with the aid of the second alternate moment equation and the distance q, which is $k^2o/r = (9.14)^2/8 = 10.45$ in. Thus:

$$[\Sigma M_a = 0] \quad 10.45\, O_t - 3.10 \times 0.707 \times (10.45 - 8) = 0; \; O_t = 0.51 \text{ lb.}$$

PROBLEMS

1. The solid cylindrical pulleys weigh 32.2 lb. each and are mounted in bearings with negligible friction. The 10 lb. force on pulley A is constant. Determine the angular acceleration α of each pulley. <u>Ans</u>. $\alpha A = 20$ rad/sec^2, $\alpha B = 12.34$ rad/sec^2.

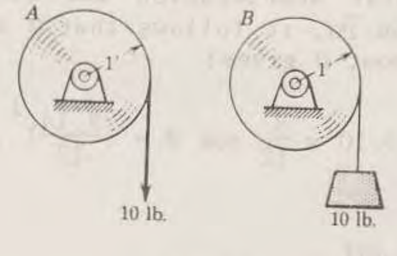


Fig.1

2. The radius of gyration of a 40 lb. flywheel about its vertical shaft is 6 in., and the center of gravity of the wheel is 0.003 in. from the axis of the shaft. If a constant moment of 10 lb.ft is applied to the flywheel through its shaft, find the components of the horizontal force F exerted on the bearing 5 sec. after the wheel starts from rest. <u>Ans</u>. $F_n = 8.05$ lb., $F_t = 0.01$ lb.

3. The rim of the flywheel shown in section is welded to the central web which in turn is welded to the hub. The weights of the web and the hub are negligible compaired with the 100 lb. rim. If the maximum safe shearing force which the rim weld and hub weld can each support is 4000 lb. per inch of weld length, find the maximum safe acceleration α which can be given to the flywheel by a torque suddenly applied to the shaft.

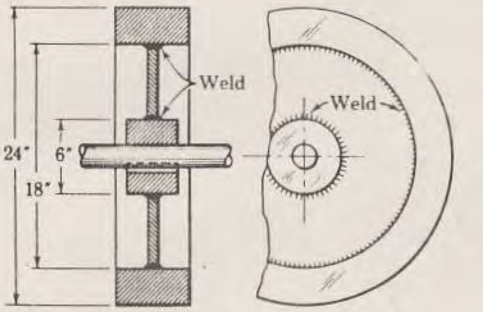


Fig. 3

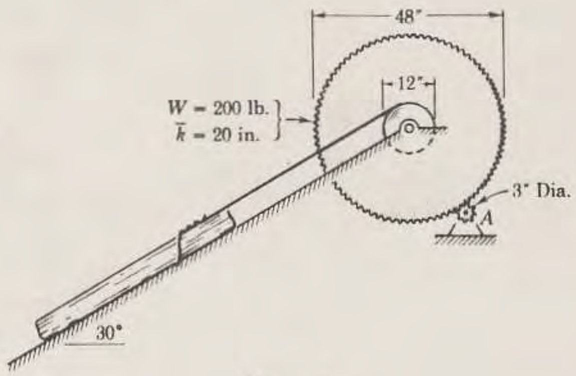


Fig. 4

4. The motor pinion A of the log hoist has a negligible moment of inertia and is subjected to a counterclockwise starting torque of 40 lb.ft. Determine the tension T in the cable if the coefficient of friction between the 500 lb. log and the incline is 0.80. Ans. T = 722 lb.

5. The gear weighs 10 lb. and has a radius of gyration of 4 in. Each rack weighs 12 lb. and slides against the smooth vertical guide. Determine the torque M required on the shaft of the gear to give an angular acceleration of 8 rad./sec^2.

Fig.5

6. The pendulum for the impact testing device weighs 75 lb. and has a radius of gyration about O of 26 in. The pendulum is designed so that the force on the bearing at O is the least possible value during impact with the specimen at the bottom of the swing. Determine the distance b. Also calculate the total force on the bearing at O an instant after the pendulum is released from rest at θ = 60 deg. Ans. b = 4.17 in., O = 38.7 lb.

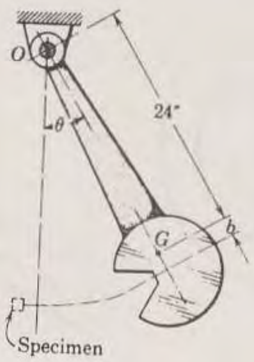


Fig.6

7. Determine the starting torque M which a motor must supply to the light pinion A in order to raise the weighted arm with an initial angular acceleration of 2 rad./sec^2. from the position shown. The radius of gyration of the 10 lb. arm assembly about its pivot is 12 in., and its center of gravity is at G. The 20 lb. rack moves with negligible friction in its guides. Ans. M = 13.4 lb.in.

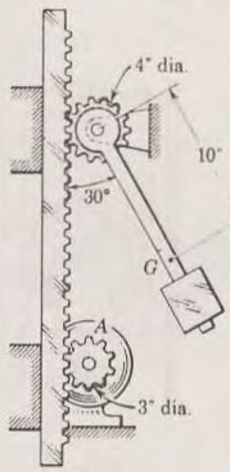


Fig.7

8. The 64.4 lb. cylindrical rotor A is mounted freely on the shaft BC. The arms D have negligible weight and are rigidly attached to the vertical shaft E with fixed axis. Through a motor drive the shaft E exerts a torque of 20 lb.ft. on the arm assembly. Determine the angular acceleration α of the arms (a) when the locking pin F is in place and (b) when the pin is removed.

<u>Ans.</u> (a) $\alpha = 8.89$ rad./sec^2., (b) $\alpha = 10$ rad./sec^2.

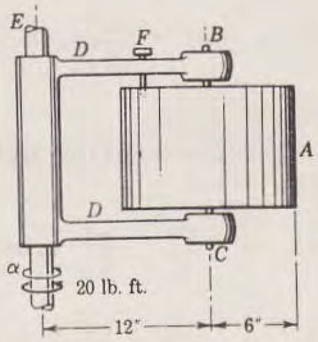
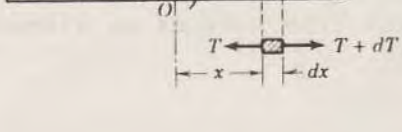


Fig.8

10. **<u>Distributed Forces in Rotation:</u>** The internal forces induced in a body by reason of high rotative speeds or appreciable accelerations are usually important design considerations. The calculation of the internal forces which result from each motions is often very involved. However, when the body is essentially uni-dimensional, such as a slender rod or a thin ring, it is usually possible to calculate the internal forces without undue difficulty. The general procedure is to isolate a differential element or a finite portion of the body with a free-body diagram and to write the motion equations for the forces acting on the part isolated.

SAMPLE PROBLEM

1. Determine the variation of the centrifugal tension in the slender rod of weight W and length 2ℓ which rotates in a horizontal plane about an axis through O normal to the rod.



Solution: The free-body diagram of an element of length dx shows a tension T acting on the left hand section, where the coordinate is x, and a tension T + dT on the right hand section, where the coordinate is x + dx. The weight of the element is assumed small compared with the tension T. If μ stands for the weight of the bar per unit length, the equation of motion for this element gives:

$$[\Sigma F_n = m r w^2] \qquad T - (T + dT) = \frac{\mu . dx}{g} xw^2,$$

$$-dT = \frac{\mu w^2}{g} x . dx.$$

At the center of the bar the tension will be designated by To, so that the integration limits give:

$$\int_{To}^{T} - dT = \frac{\mu w^2}{g} \int_{o}^{x} x . dx,$$

$$T = To - \frac{\mu w^2 x^2}{2g}.$$

Since the tension is zero at $x = \ell$, the tension at the center becomes $To = (\mu w^2 \ \ell^2)/2g$. This value for To may also be obtained by considering the acceleration of the mass center of half the bar, which is $(\frac{\ell}{2}) w^2$. Thus the force To is $\frac{1}{2}(\frac{W}{g})(\frac{\ell}{2}) w^2 = (\mu w^2 \ell^2)/2g$. The tension at any value of x becomes:

$$T = \frac{\mu w^2}{2g}(\ell^2 - x^2) = \frac{W \ell w^2}{4g} \left(1 - \frac{x^2}{\ell^2}\right) \qquad \underline{\text{Ans.}}$$

The stress at any position in the bar is the tension T divided by the cross sectional area of the bar and is known as a centrifugal stress.

PROBLEMS

1. Determine the centrifugal stress σ in the rim of a flywheel of weight density μ rotating with a constant rim speed V. Assume the radial thickness of the rim to be small compared with the radius of the wheel and neglect the effect of the web or spokes. Solve, first, by considering one half of the rim as a free body and, second, by considering the free body as an element of the rim subtending an angle dO. σ Ans. $= \frac{\mu}{g} v^2$

2. The hub and attached blades are given an angular acceleration α about the vertical shaft at O. Derive expressions for the shear force Q and bending moment M in the blade in terms of r. The blade is a uniform slender bar with a mass ρ per unit length.

Ans. $Q = \frac{\rho\alpha}{2}(R^2 - r^2)$, $M = \frac{\sigma\alpha}{6}(2R + r)(R - r)^2$

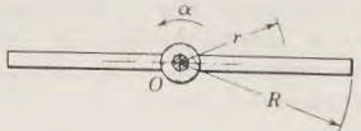


Fig. 2

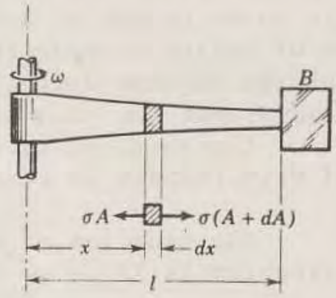


Fig. 3

3. A slender rod of mass density ρ and variable cross sectional area A is to be designed to support rotation of body B at an angular velocity w. The centrifugal force applied to the end of the rod by B divided by a desired design stress σ gives an area A_ℓ for the end section of the rod. Determine the necessary variation of the cross-sectional area A as a function of x in order that the stress be constant over the length of the rod. The free-body diagram of an element of the rod is shown.

Ans. $A = A_\ell . e^{\frac{\rho w^2}{2\sigma}(\ell^2 - x^2)}$

4. The split ring of radius r is rotating about a vertical axis through its center O with a constant angular velocity w. Use a differential element of the ring and derive expressions for the shear force N and rim tension T in the ring in terms of the angle O. Determining the bending moment Mo at point C by using one half of the ring as a free body. The mass of the ring per unit length of rim is ρ. Ans. $N = \rho r^2 w^2 \sin\theta$, $T = \rho r^2 w^2 (1 + \cos\theta)$, $Mo = 2\rho r^3 w^2$.

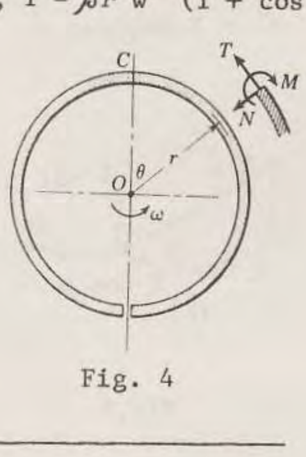


Fig. 4

11. **Work and Energy:** In previous articles, the basic equation of motion relating force, mass, and acceleration was applied to particle and rigid-body motion. In all cases attention was focused on the instantaneous relationship between the forces and the corresponding accelerations. In some problems the accelerations were integrated to obtain changes in velocities. There are many problems where the change in velocity may be determined directly from the integral of the equation of motion without first having to establish the instantaneous relationships between force, mass, and acceleration. The work-energy method developed in this article is a consequence of the first integral of the equation of motion with respect to displacement. The integral with respect to time will be discussed later.

The equation of motion for a particle of mass m moving in the x-direction is $\Sigma F_x = m\frac{d^2x}{dt^2}$. Multiplication by dx gives:

$$\Sigma F_x . dx = m dx \frac{d}{dt}\left(\frac{dx}{dt}\right) = m \frac{dx}{dt} d\left(\frac{dx}{dt}\right) = \frac{1}{2} m . d\left(\frac{dx}{dt}\right),$$

The first integral with respect to displacement is, then:

$$\int \Sigma F_x . dx = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + \text{constant}.$$

The left-hand side of the equation is the work done on m during the interval involved and can be evaluated if ΣFx is a known function of x. The right hand side is the corresponding change in the kinetic energy of m. Before applying the first integral of the equation of motion to problems it will be well to discuss the concepts of work and kinetic energy in some detail and to generalize the integral so that it may be applied to curvilinear motion and to a system composed of connected particles.

12. **Work:** The concept of work was defined as the product of a force F and the movement ds of its point of application O, Fig.4a, is:

$$dU = F . ds . \cos \alpha .$$

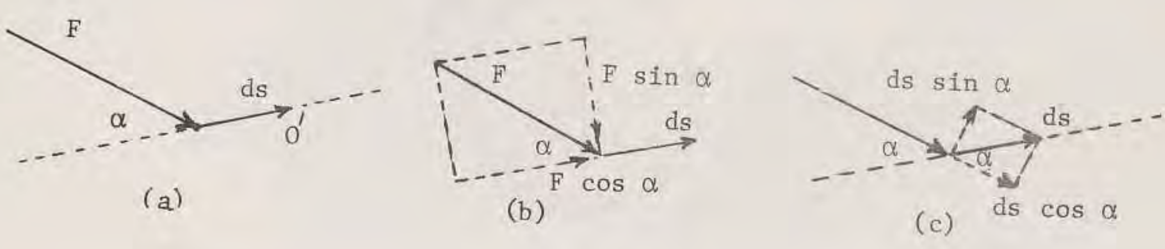


Fig.4

This definition may be viewed either as the force component F cos α in the direction of the displacement times the displacement, Fig. 4b., or as the force times the displacement component ds. cos α in the direction of the force, Fig.4c. With this definition it is seen that the component of the force at right angles to the displacement does no work. The work done by the force is positive when the working component F cos α has the same sense as the displacement and negative when in the opposite sense. Work is a scalar quantity with the dimensions of [distance] x [force].Work and moment are dimensionally the same, and in order to distinguish between them work may be expressed as feet pounds [ft.lb.] and moment as pounds feet (lb.ft.). The fundamental difference between work and moment is that work is a scalar involving the product of force and distance both measured in the same direction, whereas moment is a vector and is the product of force and distance measured at right angles to the force.

During a finite movement the work done by the force is:

$$U = \int F \cos \alpha \, ds.$$

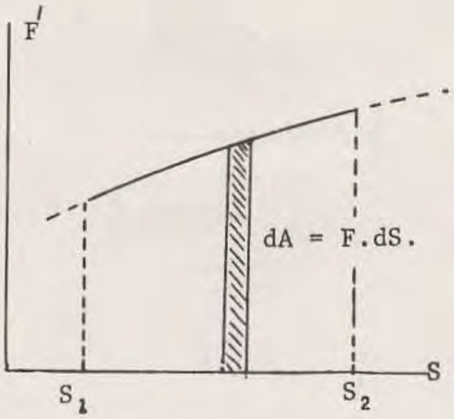


Fig.5

This expression can be integrated if the relation between F and S and between cos α and S is known.

Experimental data often enable a graph to be constructed of the working component $F' = F \cos \alpha$ and the displacement S of its point of application as schematically represented in Fig.5. The different area dA under the curve during the movement ds is the work done by F' during that interval, and the net or total work done between any two displacements S_1 and S_2 is the area under the curve between these limits.

In the case of an elastic spring of stiffness k and negligible weight the force F supported by the spring at any deformation x, either compression or extension, is $F = kx$. Thus the work done on a spring during a compression or extension from its undeformed position, Fig.6, is:

$$\int F.dx = \int_{x1}^{x2} kx.dx = \frac{1}{2} kx_2^2 - \frac{1}{2} kx_1^2.$$

which represents the trapezoidal area on the F-x diagram. It is noted that positive work is done on the spring when it is being extended or compressed. In the reverse process during a release from its tension or compression, negative work is done on the spring which means that the spring does positive work on the body against which it is allowed to act.

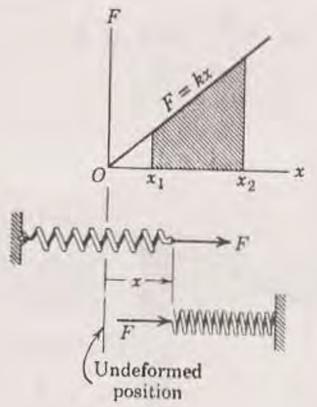


Fig.6

The work done by a couple M acting on a body, Fig.7, during a rotation dθ of the body in the plane of the couple is $dU = M.d\theta$. This expression is easily obtained by representing the couple by two forces and evaluating the work done by each force during the rotation dθ. The total work done by M during a finite angular displacement θ is:

$$U = \int M.d\theta.$$

The angle θ is expressed in radian measure, and work is positive when the rotation is in the sense of the couple and negative when in the sense opposite to the couple. A couple does no work during a movement which is entirely one of translation since the angular displacement is zero. In the event that the body rotates in a plane other than the plane of the couple the work done equals the magnitude of the couple vector times the magnitude of the rotation vector multiplied by the cosine of the angle between the vectors.

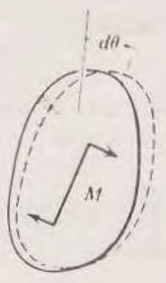


Fig.7

When a body slides on a fixed surface, the work done by the friction force acting on the body is negative since the friction force acts in the direction opposite to the displacement. In the case of a wheel which rolls on a fixed surface without slipping, a static friction force acts and does no work since the point of application does not slip. If the wheel slips as it rolls, kinetic friction is generated and negative work is done on the wheel.

The total work done by any system of forces and couples acting on a body during any movement is the algebraic sum of the works done by each force and couple considered separately.

13. **Kinetic Energy of a Particle:** The first integral of the equation of motion as given in Art.11 will now be redelivered for the motion of a particle of mass m along any curved path, Fig.8. The resultant of all forces acting on m is F. Only the tangential component of F does work on m during motion along the path. During a displacement ds the work done on m is:

$$dU = F \cos \alpha ds.$$

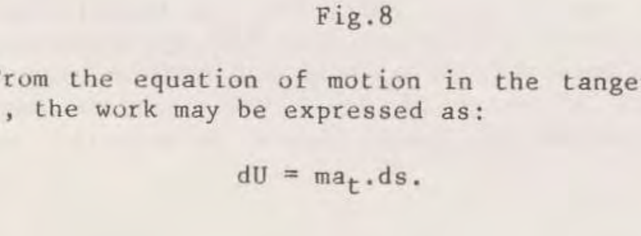


Fig.8

From the equation of motion in the tangential direction, $F \cos \alpha = ma_t$, the work may be expressed as:

$$dU = ma_t.ds.$$

But $a_t.ds = v.dv$, so that:

$$dU = mv.dv.$$

The net work done on m during an interval of its motion for which the velocity changes from Vo to V is:

$$\Delta U = \ dU = \int_{vo}^{o} mv.dv = \frac{1}{2}mv^2 - \frac{1}{2}mv^{\ \ 2}.$$

The term mv is the work done in bringing the particle from rest to a velocity V. and is known as the kinetic energy T of the particle. Hence the definition:

$$T = \frac{1}{2}mv^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

The net work done on m in changing its velocity from Vo to V therefore equals the change T-To in its kinetic energy, or:

$$\Delta U = \Delta T \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (15)$$

Equation(15) is known as the Work-Energy Equation and always involves the change in kinetic energy as a result of the net work done during the corresponding interval of motion.

If a particle moving with a velocity V is allowed to act on some body and is brought to rest during this action, the loss of its kinetic energy equals the work done on the other body by the contact force. Thus kinetic energy represents the capacity to do work by reason of acquired velocity.

Kinetic energy is a scalar quantity which depends only on the mass and the magnitude of the velocity. Since the velocity V is squared, kinetic energy is always a positive quantity. The units of kinetic energy are the same as those of work, as may be seen from the dimensional equation:

$$[\frac{1}{2}mv^2] = [FL^{-2}T^2][LT^{-1}]^2 = [LF]$$

When a body is subjected to a system of forces the resultant of which acts through the center of mass, the body may be treated as a particle. Thus the work-energy equation as developed for a particle may be applied to such a body.

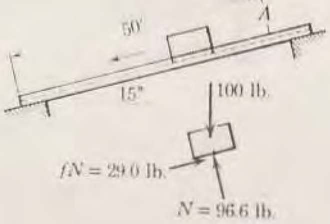
When two or more particles (or translating bodies considered as particles) are joined together by connections which are frictionless and incapable of elastic deformation, the forces in the connections occur in pairs of equal and opposite forces, and the points of application of these forces necessarily have identical movements. Hence the net work done by these internal forces is zero during any movement of the system. Thus Eq.(15) is applicable to the entire system, where ΔU is the total or net work done on the system by external forces and ΔT is the change in the total kinetic energy of the system. The total kinetic energy is the algebraic sum of the kinetic energies of all elements of the system.

The method of work has a basic advantage over the force and moment summation method for solving the equilibrium problem for a system of multiconnected bodies. The same advantage exists in the method of work and energy for the dynamics of interconnected bodies since again consideration of internal forces is not necessary.

Application of the work-energy method calls for an isolation of the body or system under consideration. For a single body a free-body diagram showing all externally applied forces should be drawn. For a system of connected bodies without springs an active-force diagram which shows only those external forces which do work (active forces) on the system may be drawn.

SAMPLE PROBLEMS

1. Determine the velocity V of the 100 lb. crate when it reaches the bottom of the chute if it is given an initial velocity of 15 ft./sec. down the chute at A. The coefficient of friction is 0.30.



<u>**Solution:**</u> The free body diagram of the crate is drawn and includes the normal and friction forces calculated in the usual manner. The work done by the component of the weight down the plane is positive, whereas that done by the friction force is negative. The total or net work done during the interval is, then:

$$\Delta U = (100 \sin 15° - 29.0)\ 50 = -155 \text{ ft.lb.}$$

The change in kinetic energy is:

$$\Delta U = \frac{1}{2}\cdot\frac{100}{32.2}(v^2 - 225)$$

The work-energy equation gives

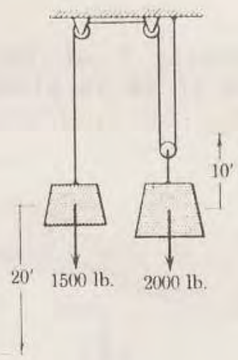
$$[\Delta U = \Delta T] \quad -155 = \frac{1}{2}\cdot\frac{100}{32.2}(v^2 - 225)$$

$$v^2 = 125; \quad v = 11.2 \text{ ft.sec.} \quad \underline{\text{Ans.}}$$

---

2. In the system shown the weights of the cable and pulleys and the friction in the pulley bearings are negligible. Determine the velocity V of the 2000 lb. weight after it has moved 10 ft. from the rest position from which the system was released.

Solution: The only external forces which do work on the entire system are the weights of the two bodies. With these 2 forces indicated the sketch may be used as the active-force diagram.



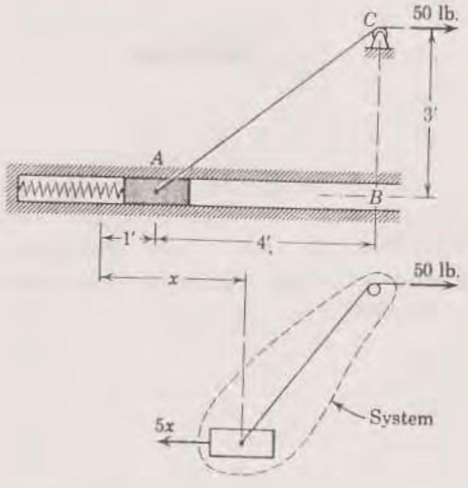
The 10 ft. displacement of the 2000 lb. weight is clearly up, whereas that of the 1500 lb. weight is 20 ft. down. Also the velocity of the 1500 lb. weight is twice that of the 2000 lb. weight. The work-energy equation for the system is applied and gives:

$$[\Delta U = \Delta T] : \quad -2000 \times 10 + 1500 \times 20 = \frac{1}{2}\cdot\frac{2000}{32.2}(v^2 - 0) + \frac{1}{2}\cdot\frac{1500}{32.2}(4v^2 - 0),$$

$$v = 8.97 \text{ ft./sec.} \qquad \underline{\text{Ans.}}$$

---

3. The 20 lb. slider starts from rest at A with an initial tensile force of 5 lb. in the attached spring and moves in the smooth horizontal slot under the action of a constant 50 lb. force in the cable. If the modulus of the spring is 5 lb./ft., determine the velocity V. of the block as it passes the position B.



Solution: It will be assumed that the stiffness of the spring is small enough to allow the block to reach position B. The active-force diagram for the system composed of the block and the cable is shown for a general position. The spring force and the 50 lb. tension are the only forces external to this system which do work on the system. The force of the guide on the block, the weight of the block, and the reaction of pulley on the cable do no work on the system and are not included on the active-force diagram.

The displacement x of the block is measured from the undeformed position of the spring which is a distance $F/k = 5/5 = 1$ ft. to the left of A. The work done on the system by the spring force during the movement from A to B is negative and is :

$$- \int_1^5 5x.dx = -\frac{1}{2} \times 5 \ (5^2 - 1^2) = -60 \text{ ft.lb.}$$

If the displacement x had been measured from the starting position of the block, the spring force would be $5(x+1)$, and the limits of integration would be 0 and 4.

The work done on the system by the constant 50 lb. force in the cable is the force times the net horizontal movement of the cable over pulley C which is $\sqrt{4^2 + 3^2} - 3 = 2$ ft. Thus the work done is $50 \times 2 = 100$ ft.lb., and the work-energy equation applied to the system gives:

$$[\Delta U = T] \qquad -60 + 100 = \frac{1}{2}\cdot\frac{20}{32.2}\,(v^2 - 0),$$

$$v = 11.35 \text{ ft./sec.} \qquad \underline{\text{Ans}}.$$

PROBLEMS

1. A projectile weighing 8 lb. is fired through a stock of asbestos sheets 6 ft. thick. If the projectile approaches the asbestos with a velocity of 1800 ft./sec. and emerges with a velocity of 900 ft/sec., determine the average penetration resistance R over the 6 ft.
Ans. R = 50,300 lb.

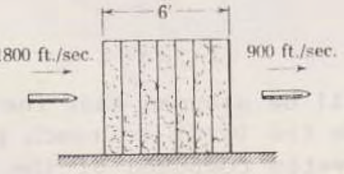


Fig.1

2. A small 2 oz. bead starts from rest at A and slides freely in the vertical plane along the fixed wire under the action of the constant 4 oz. horizontal force. Find the velocity V of the bead as it hits the stop at B.
Ans. V = 1390 ft./sec.

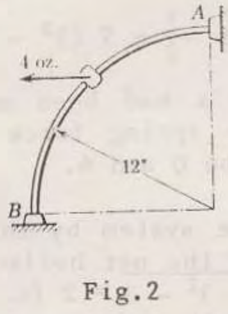


Fig.2

3. A car is travelling at 30 mi./hr. down a 5 per cent grade when the brakes on all 4 wheels lock. If the kinetic coefficient of friction between the tires and the road is 0.70, find the distance S which the car skids in coming to a stop. Ans. $S = 46.3$ ft.

4. Find the total energy absorbed (negative work) by the brakes of a 1200 ton passenger train in bringing it to a stop from a velocity of 60 mi./hr. in a distance of 2 mi. down a 1 per cent grade.
Ans. $E = 542 \times 10$ ft.lb.

5. The weight is released from rest with the cord in the horizontal position shown. When the bottom position is reached, the cord strikes the small fixed bar shown in section at A, and the weight follows the dotted path. Calculate the velocity of the weight at B and C.

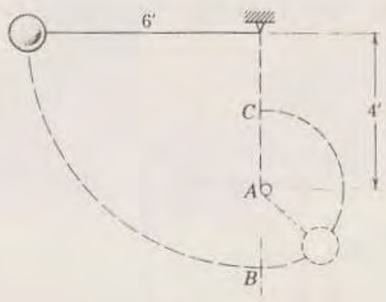


Fig. 5

6. The slider weighs 30 lb. and is constrained to slide on the fixed circular bar. If the slider is elevated from rest at A to position B by a constant 50 lb. force in the cable and if friction is negligible, determine the velocity v of the slider as it reaches B.

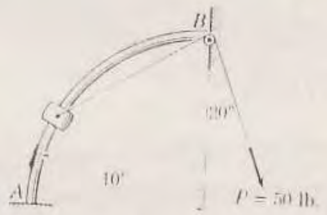


Fig.6

7. The weight W is attached to an elastic cable, and the lower supports are raised slightly until the cable tension and elongation are zero. If the supports are suddenly removed, find the maximum tension T in the cable . Ans. $T = 2W$.

Translation: The translating body of mass m in Fig.9(a) has a velocity v. The kinetic energy of any particle of mass m_i is $\frac{1}{2} m_i v^2$, and the kinetic energy of the entire body is:

$$T = \Sigma \frac{1}{2} m_i v^2 = \frac{1}{2} v^2 \; \Sigma m_i ,$$

$$\text{or} \quad T = \frac{1}{2} mv^2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \; (16)$$

Fixed-Axis Rotation: The body in Fig.9(b) is rotating about a fixed axis through O with an angular velocity w. The linear velocity of any particle of mass m is rw, and the kinetic energy of this particle is $\frac{1}{2} m_i (rw)^2$. The kinetic energy of the entire rotating body is the sum of the kinetic energies of all its particles and is:

$$T = \Sigma \frac{1}{2} m_i r^2 w^2 = \frac{1}{2} w^2 \; \Sigma m_i r^2$$

$$\text{or} \quad T = \frac{1}{2} I_O w^2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \; (17)$$

The term Io is the moment of inertia about the fixed axis of rotation. The similarity between the expressions for kinetic energy of rotation and translation should be noted. Moment of inertia and angular velocity replace mass and linear velocity, respectively. The dimensions of both expressions are identical and can be verified easily.

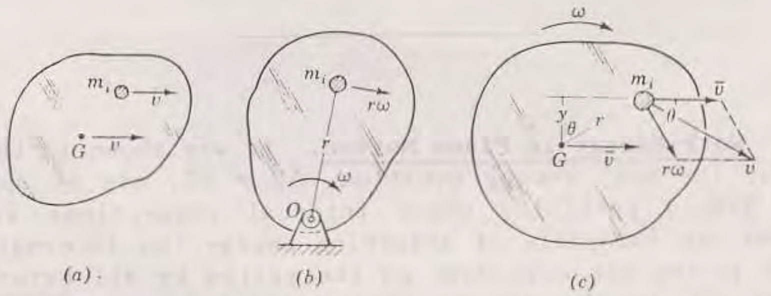


Fig.9

Plane Motion: The body in Fig.(9c) has any plane motion, and, at the instant considered, the velocity of its mass center G is $\bar{v}$ and the angular velocity of the body is w. The velocity v of any particle is conveniently expressed in terms of $\bar{v}$ and the velocity rw of the particle relative to G as shown. The square of the particle velocity is obtained by the law of cosines and is:

$$v^2 = \bar{v}^2 + r^2 w^2 + 2\bar{v} r w \cos \theta$$

The kinetic energy of the representative particle of mass m_i is $\frac{1}{2} m_i v^2$, and that for the entire body is:

$$T = \Sigma \frac{1}{2} m_i \bar{v}^2 + \Sigma \frac{1}{2} m_i r^2 w^2 + \Sigma (\frac{1}{2} m_i) (2\bar{v}rw \cos 0),$$

$$= \frac{1}{2} \bar{v}^2 \Sigma m_i + \frac{1}{2} w^2 \Sigma m_i r^2 + \bar{v}w\Sigma m_i y.$$

The last summation is zero since the y-coordinate to the center of gravity is zero, and the second summation is merely the moment of inertia $\bar{I}$ about the center of gravity G. Thus the kinetic energy for any rigid body having plane motion is:

$$T = \frac{1}{2} m\bar{v}^2 + \frac{1}{2} \bar{I}w^2 \qquad \text{............... (18)}$$

This expression clearly shows the separate contribution to the total kinetic energy due to the translational velocity of the mass center and the rotational velocity about the mass center.

The kinetic energy of plane motion may be expressed also in terms of the rotational velocity about the instant center C of zero velocity. Since C momentarily has zero velocity, the proof for Eq.(17) holds equally well for this point. Thus the kinetic energy of plane motion may be expressed by:

$$T = \frac{1}{2} I_c w^2 \qquad \text{.............. (19)}$$

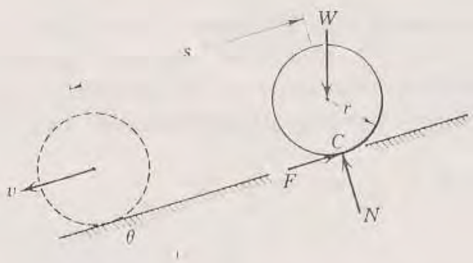
in place of Eq.(18).

The total kinetic energy T of a system of bodies having plane motion is the sum of the kinetic energies of all its parts calculated by Eqs.(16), (17), (18), or (19) for the particular kinds of motion involved. When the work-energy principle $\Delta U = \Delta T$ is applied to such a system, ΔT is the change in the total kinetic energy of the system and ΔU is the net work done by all active forces which are applied externally to the system during the interval involved.

In applying the work-energy principle to a single rigid body in plane motion either a free-body diagram or an active force diagram may be drawn. When applying the work-energy principle to "a connected system of rigid bodies, an active-force diagram of the entire system" should be drawn to isolate the system and disclose all external forces which do work on the system.

SAMPLE PROBLEMS

1. Determine the velocity v of the center of the circular disk after it has rolled a distance S down the incline from rest. Friction is sufficient to prevent slipping.



Solution: Of the 3 forces shown on the free-body diagram of the disk only the weight does work. The friction force does no work if the wheel does not slip.

Thus: $\Delta U = W.S \sin \theta,$

$$\Delta T = \frac{1}{2}\left(\frac{W}{g}\right) v^2 + \frac{1}{2}\left(\frac{1}{2}\frac{W}{g} r^2\right) w^2 = \frac{3}{4}\frac{W}{g} v^2$$

The work-energy equation is:

$$[\Delta U = \Delta T] \qquad W.S. \sin \theta = \frac{3}{4}\frac{W}{g} v^2; \quad v = \frac{\sqrt{4gS \sin \theta}}{3}$$

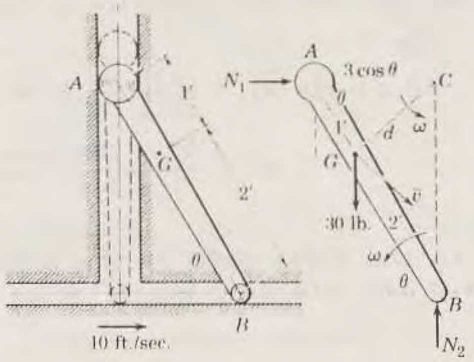
This kinetic energy may also be expressed by:

$$\frac{1}{2} I_C w^2 = \frac{1}{2}\left(\frac{1}{2}\frac{W}{g} r^2 + \frac{W}{g} r^2\right) w^2 = \frac{3}{4}\frac{W}{g} v^2$$

---

2. The 30 lb. link with center of gravity at G has a centroidal radius of gyration of 0.8 ft. and moves in the vertical plane under the action of its own weight. The ends of the link are confined to move in the horizontal and vertical slots. If end B is given a horizontal velocity of 10 ft./sec. when the link passes the dotted vertical position, compute the linear velocity v of end A as the link reaches the horizontal position by analyzing the initial and final conditions directly. Also determine the expression for the angular velocity w of the link at any instant when its angle with the horizontal is θ.

Solution: The free-body diagram shows that the 30 lb. weight is the only force which does work on the body. Between the vertical and horizontal positions of the link point G drops through a vertical distance of 2 ft. so that $\Delta U = 30 \times 2 = 60$ ft.lb.



In the vertical position end A is the instant center of zero velocity, so that the kinetic energy for the initial position is $T = \frac{1}{2} I_A W_1^2$. In the final position end B is the instant center of zero velocity, and the kinetic energy is $T_2 = \frac{1}{2} I_B W_2^2$. (The kinetic energy may also be computed from $T = \frac{1}{2} mv^2 + \frac{1}{2} \bar{I}w^2$ by relating v and w) Thus:

$$T_1 = \frac{1}{2} Ic\, w^2 = \frac{1}{2} [mk^2 + md^2] \left[\frac{v}{r}\right]^2 = \frac{1}{2} \left[\frac{30}{g} (0.8^2 + 1^2)\right] \left[\frac{10}{3}\right]$$

$$= 8.49 \text{ ft.lb.}$$

and $$T_2 = \frac{1}{2} \left[\frac{30}{g} (0.8^2 + 2^2)\right] \left[\frac{v^2}{3}\right] = 0.24\, v^2 \text{ ft.lb.}$$

Application of the work-energy principle to the entire interval gives:

$[\Delta U = \Delta T]$ $\qquad 60 = 0.24\, v^2 - 8.49; \qquad v = 16.89$ ft./sec. Ans.

For the general position θ the center of gravity G has dropped a distance of $(2-2 \sin \theta)$, so that the work done is $\Delta U = 60(1-\sin \theta)$ ft.lb. The kinetic energy at this position is $T = \frac{1}{2} m\bar{v}^2 + \frac{1}{2} \bar{I}w^2$ where w is the angular velocity of the body. If the instant center C of zero velocity is used, $\bar{v} = wd$ where w is also the angular velocity of line GC which rotates instantaneously with the body about C. Thus:

$$T = \frac{1}{2} mw^2 d^2 + \frac{1}{2} m\bar{k}^2 w^2 = \frac{1}{2} m (d^2 + \bar{k}^2)\, w^2 \frac{1}{2} Icw^2.$$

From the law of cosines $d^2 = 1^2 + 3^2 \cos^2\theta = 1 + 3\cos^2\theta$, and

$$T = \frac{1}{2}\cdot\frac{30}{32.2}(1 + 3\cos^2\theta + 0.8^2)\, w^2 = \frac{15}{32.2}(1.64 + 3\cos^2\theta)\, w^2.$$

Finally the work-energy principle for the interval between the vertical position and the inclined position gives:

$$[\Delta U = \Delta T] \quad 60\,(1-\sin\theta) = \frac{15}{32.2}(1.64 + 3\cos^2\theta)\, w^2 - 8.49$$

from which:

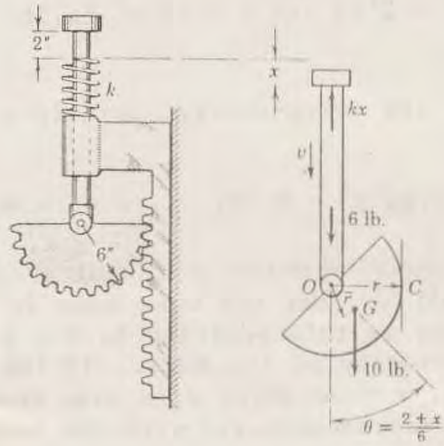
$$w^2 = \frac{32.2\,(68.49 - 60\sin\theta)}{15\,(1.64 + 3\cos^2\theta)} \qquad \underline{\text{Ans.}}$$

The answer to the first part of the problem may be checked by computing w for $\theta = 0$ and obtaining the velocity of B from $v = 3w$.

---

3. Specify the necessary modulus k of the spring which will allow the gear sector to rotate a maximum of $\frac{1}{8}$ turn when released from rest in the position shown. Also find the velocity v of the 6 lb. plunger when the spring is compressed 1 in. during this motion. The gear sector weighs 10 lb. and may be treated as a semi-circular disk. Also, friction in the parts is negligible.

Solution: The sector and attached plunger are taken as the system to be isolated, and the active-force diagram, showing all external forces which act on the system and do work, is drawn. The 10 lb. and 6 lb. weights and the spring force are the only such active forces.



For a rotation of 1/8 of a turn $\theta = \frac{\pi}{4}$, and the compression of the spring is $x = \frac{6\pi}{4} - 4 = 2.71$ in. Also $\bar{r} = \frac{4\pi}{3\pi} = 2.55$ in. The vertical movement of G for $\theta = \frac{\pi}{4}$ is:

$$(2 + x) - (\bar{r} - \bar{r}\cos\theta) = 4.71 - 2.55\ (1 - 0.707) = 3.96 \text{ in.}$$

The work-energy principle for $\Delta T = 0$ is:

$$[\Delta U = 0] \qquad (10 \times 3.96) + (6 \times 4.71) - \frac{1}{2} k\ (2.71)^2 = 0,$$

$$k = 18.48 \text{ lb. } \underline{\text{Ans.}}$$

For a spring compression of 1 in. the angular movement of the sector is:

$$\theta = \frac{2 + 1}{6} = 0.5 \text{ rad. or } \theta = 28° \ 39'$$

and the vertical movement of G is:

$$(2 + 1) - 2.55\ (1 - 0.878) = 2.69 \text{ in.}$$

Thus the net work done during the 1 in. compression of the spring is:

$$\Delta U = (10 \times 2.69) + (6 \times 3) - (\frac{1}{2} \times 18.48 \times 1^2) = 35.7 \text{ in.lb.}$$

The kinetic energy of the disk may be computed from the expression $\frac{1}{2} I_c w^2$, where:

$$I_c = \bar{I} + m\,\overline{GC}^2 = I_o - M\bar{r}^2 + m\,\overline{GC}^2 + I_o + m\,(\overline{GC}^2 - \bar{r}^2).$$

The law of cosines applied to the triangle OGC gives:

$$\overline{GC}^2 - \bar{r}^2 = r^2 - 2r\bar{r}\cos(\frac{\pi}{2} - \theta) = 21.3 \text{ in.}^2,$$

and the moment of inertia about O is:

$$I_o = \frac{1}{2}\,(\frac{1}{2} \times 2\ mr^2) = \frac{1}{2}\,mr^2 = \frac{10 \times 6}{2 \times 32.2 \times 12} = 0.466 \text{ lb.in.sec.}^2$$

Therefore the kinetic energy of the disk is:

$$\frac{1}{2}\,I_c w^2 = \frac{1}{2}\,(0.466 + \frac{10}{32.2 \times 12} \times 21.3)\,\frac{v^2}{6^2} = 0.01414\ v^2 \text{ in.lb.}$$

Finally, the kinetic energy of the plunger is accounted for, and the work-energy equation gives:

$$[\Delta U = \Delta T] \qquad 35.7 = 0.01414\ v^2 + \frac{1}{2}\cdot\frac{6}{32.2 \times 12}\,v^2,$$

$$v = 40.4 \text{ in./sec.} \qquad \underline{\text{Ans.}}$$

The kinetic energy of the gear sector may be determined also from the relation $\frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}w^2$. In this method it is necessary to relate $\bar{v}$ and w to the velocity v of the plunger, and the labour is comparable to that of obtaining Ic as was done in the present solution.

PROBLEMS

1. The suspended log shown is used as a battering ram. At what angle θ should the log be released from rest in order to strike the object to be smashed with a velocity of 20 ft./sec.?

Ans. θ = 41° 16'

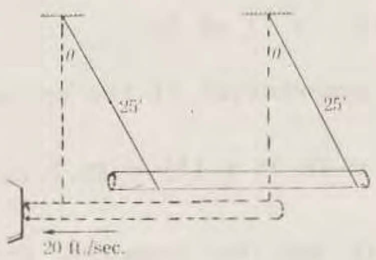


Fig. 1

Fig. 2

2. Find the torque M applied to the 40 lb. circular disk necessary to give its center a velocity of 4 ft./sec. in a distance of 10 ft. up the incline from rest. The wheel does not slip.

Ans. M = 5.47 lb.ft.

3. The gear and attached drum have a combined weight of 150 lb. and a radius of gyration of 10.5 in. The weight of the motor pinion is small and may be neglected. The 400 lb. load acquires an upward velocity of 15 ft./sec. after rising 20 ft. from rest with constant acceleration. Determine the torque M on the motor pinion.

Ans. M = 52.5 lb.ft.

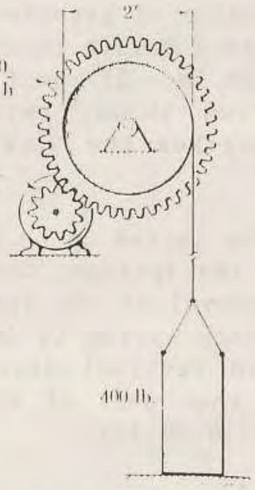


Fig. 3

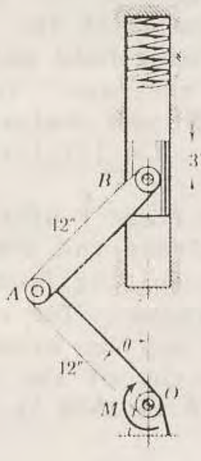


Fig. 4

4. Each of the two symmetrical toggle links weighs 10 lb. and has a centroidal radius of gyration of 4 in. The slider at B weighs 5 lb. and moves freely in the vertical guide. The spring modulus is 4 lb./in. If a constant torque M of 250 lb.in. is applied to link OA through its shaft at O starting from the rest position at θ = 45°, determine the angular velocity w of OA at the instant when θ = 0.
Ans. w = 7.36 rad./sec.

5. The 100 lb. load A and the attached 50 lb. sheave B have an upward velocity of 10 ft./sec. after rising 15 ft. from rest under the action of the constant 85 lb. cable tension. Determine the frictional moment Mf in the bearing O if the radius of gyration of the sheave is 12 in. and the cable does not slip on the sheave. Friction along the vertical guides is negligible. Ans. Mf = 3.25 lb.ft.

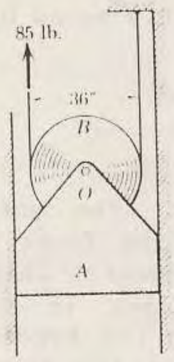


Fig. 5

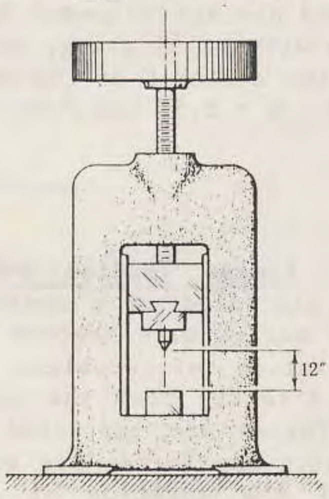


Fig. 6

6. A type of press frequently used in the hot working of nonferrous metals is shown in the figure. The flywheel and attached screw together weigh 600 lb. and have a radius of gyration about the vertical center line of 16 in. The screw has a double thread of 2 in. lead, and the forming head and die weigh 80 lb. If the flywheel has a speed of 100 rev./min. in the position shown, determine the rotational speed N just before the die strikes the work. Neglect friction. Ans. N = 117.1 rev./min.

7. If the frame F of the car must be jacked up to a height of x = 14.5 in to release the compression in the springs, determine the least value of x resulting from a sudden removal of the jack if there are no shock absorbers. The constant of each spring is 409 lb./in., and the line AC may be assumed to remain vertical throughout the movement. The weight of the frame F and that part of the attached body supported by F is 1500 lb. Ans. x = 4.68 in.

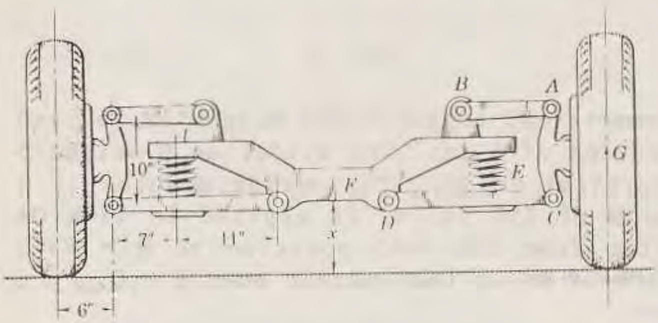


Fig. 7

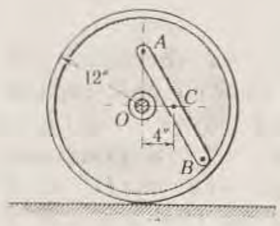


Fig. 8

8. The bar ACB weighs 10 lb. with center of gravity at C and has a centroidal radius of gyration of 6 in. The bar is screwed to the face of the wheel as shown. The wheel weighs 20 lb. with center of gravity at O and has a centroidal radius of gyration of 0.8 ft. If the wheel and bar are released from rest in the position shown and the wheel rolls without slipping, determine the angular velocity w of the wheel when the center C of the bar is directly below O.

Ans. w = 2.32 rad./sec.

---

15. **Linear Impulse and Momentum:** The work-energy method previously discussed is a consequence of the first integral of the equation of motion with respect to displacement. This approach leads to the solution of problems where changes in velocity may be calculated directly from the work done by the forces on the system where the forces are expressed in terms of the displacements. For problems where the forces are expressed in terms of the time a first integral of the equation of motion with time is implied. This approach leads to the quantities of impulse and momentum.

The equation of motion of a particle of mass m in the x-direction is:

$$\Sigma Fx = md_x = m\frac{dv_x}{dt},$$

where Fx is the sum of the x-components of all forces acting on m, and a_x and Vx are the x-components of the acceleration and velocity, respectively. Since the mass m of the single particle is a constant, this equation may be written in either of the two forms:

$$\Sigma Fx = \frac{d}{dt}(mv_x) \quad \text{or} \quad \Sigma Fx.dt = d\,(mv_x) \;\ldots\ldots\ldots\ldots\ldots\; (20)$$

The product of mass and linear velocity is defined as "linear momentum", and thus the first of Eqs.(20) states that "The resultant force in any one direction on a particle of mass m equals the time rate of change of its linear momentum in that direction". This formulation is an alternate way of stating Newton's second law of motion. "The product of force and time is defined as Linear Impulse, and thus the second of Eqs.(20) states that the linear impulse of ΣFx on m during time dt. equals the change in linear momentum. The dimensions of both linear impulse and linear momentum are [Force] x [Time], (lb.sec.). The relations expressed by Eqs.(20) may also be written for the y- and z- directions, and these equations are, then, the scalar components of a single sector relation which may be stated as:

$$\Sigma F = \frac{d}{dt}(mv) \quad \text{or} \quad \Sigma.F.dt = d\,(mv).$$

In the vector formulation ΣF is the resultant of all forces on m, and mv is the resultant linear momentum of m. Linear inpulse and linear momentum are both vector quantities which have the directions of ΣF and v, respectively. Thus the direction of ΣF coincides with the direction of the change in linear momentum which is also the direction of the acceleration.

The action of ΣFx during a finite interval of time t is given by integration of the second of Eqs.(20). The integral is:

$$\int_0^t \Sigma F_x.dt \;=\; mv_x - mvo_x, \;\ldots\ldots\ldots\ldots\ldots\; (21)$$

where vx is the velocity in the positive x-direction at time t and vo_x is the velocity in the positive x-direction at time t=0. The integral on the left side of the equation is the linear impulse of ΣFx during the time t, and the right side of the equation is the corresponding change in the linear momentum. When the functional relation between ΣFx and t is unknown but experimental data for the variation of ΣFx with t are available, the total impulse may be found by approximating the area under the curve of ΣFx versus t. If ΣFx is constant, the expression becomes:

$$\Sigma Fxt \;=\; mv_x - mvo_x.$$

It was known that the resultant of all forces acting on a translating rigid body passes through the center of mass and that such motion may be analyzed as though the body were a particle. Thus the foregoing principles of impulse and momentum developed for the motion of a particle may be applied equally well to a translating rigid body.

Consider now any general system of particles. These particles need not be joined but may be considered to have any motion whatsoever. If F_{1_x}, F_{2_x}, F_{3_x}, ... represent the x-components of all forces applied to a representative particle of mass m_i from sources external to the system, and if f_{1_x}, f_{2_x}, f_{3_x}, represent the x-components of all internally applied forces on m_i, the first of Eqs.(20) may be written:

$$F_{1_x} + F_{2_x} + F_{3_x} + \dots\dots\dots + f_{1_x} + f_{2_x} + f_{3_x} + \dots\dots\dots = \frac{d}{dt} (m_i v_{i_x}),$$

where v_{i_x} is the x-component of the velocity of m_i. Similar expressions may be written for each particle of the system. By adding all these equations and remembering that the sum of the internal actions and reactions is zero, the sum of external forces in the x-direction becomes:

$$\Sigma Fx = \Sigma \frac{d}{dt} (m_i v_{i_x}) = \frac{d}{dt} \Sigma (m_i v_{i_x})$$

The expression $\Sigma(m_i v_{i_x})$ is the sum of the linear momental of all particles in the x-direction and is defined as the linear momentum Gx of the system in the x-direction. This momentum may be expressed in terms of the motion of the mass center. The principle of moments is:

$$m\bar{x} = \Sigma (m_i x_i),$$

where x_i is the x-coordinate of m_i, m is the total mass of the system, and $\bar{x}$ is the x-coordinate to the mass center. If attention is focused on the total system, its mass may be considered as constant, and differentiation with respect to time gives:

$$m\bar{v}_x = \Sigma (m_i v_{i_x}) = G_x.$$

Thus the linear momentum of any given system equals the total mass of the system multiplied by the velocity of the center of mass.

The symbol G_x and its interpretation may now be used in the equation for ΣF_x. With similar expressions for the y- and z-directions there result:

$$\Sigma Fx = \frac{d\ G_x}{dt}$$

$$\Sigma Fy = \frac{d\ G_y}{dt} \qquad \text{.............. (22)}$$

$$\Sigma Fz = \frac{d\ G_z}{dt}$$

Equations(22) are the scalar components of the single vector equation:

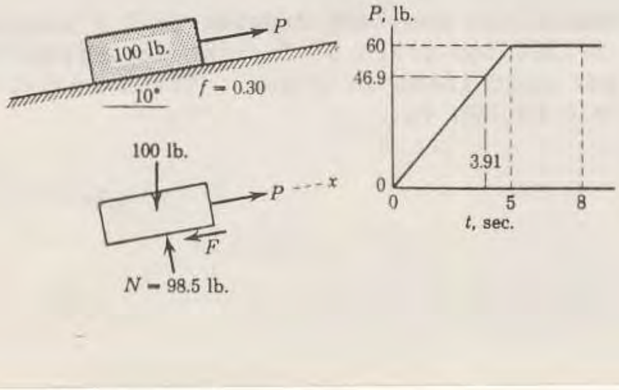
$$\Sigma F = \frac{dG}{dt} \qquad \text{.............. (23)}$$

where $G = m\bar{v}$. Equation(23) states that the resultant of the external forces acting on any given system of particles equals the magnitude and direction of the time rate of change of the linear momentum of the system. This formulation is an extremely important concept in mechanics, and is an alternate way of expressing the equation of motion of the mass center of any system.

Application of the principles of impulse and momentum requires the use of a free-body diagram where all external forces acting on the body on system are accounted for. In the method of work and energy it is necessary to consider only these forces which do work, whereas with the method of impulse and momentum all forces with components in the directions of motion exert impulses whether they do work or not. The generality of Eq.(23), derived for any given system of particles, is not needed when dealing with a translating rigid body. This generality will be found useful in discussing problems which involve mass flow in articles which follow.

SAMPLE PROBLEM

1. The 100 lb. weight is pulled up the incline from rest by the action of the force P, which varies with the time according to the accompanying graph. Determine the velocity v of the weight 8 sec. after P begins to act. The coefficient of friction is 0.30.



Solution: The free-body diagram of the block is drawn and discloses three forces which have components in the direction of motion. There is zero resultant force on the 100 lb. weight until P overcomes the limiting frictional force of $0.30 \times 100 \cos 10° = 29.5$ lb. and reaches the value of:

$$P = 29.5 + 100 \sin 10° = 46.9 \text{ lb.}$$

when $t = \frac{(46.9)}{60} 5 = 3.91$ sec. For the remaining 4.09 sec. the net force in the direction of motion is:

$$\Sigma Fx = P - 46.9$$

The impulse during the 4.09 sec. is:

$$\int_{3.91}^{8} P.dt - (46.9 \times 4.09) = 46.4 \text{ lb.sec.}$$

where the integral is discontinuous and may be obtained from the area under the curve to the right of $t = 3.91$ sec. The impulse-momentum principle gives:

$$\left[\int \Sigma Fx.dt = \Delta\,(mv_x)\right] : \qquad 46.4 = \frac{100}{32.2}(v - 0),$$

$$v = 14.94 \text{ ft./sec.} \qquad \underline{\text{Ans}}.$$

PROBLEMS

1. A jet-propelled airplane weighing 8 tons (16000 lb.) is flying at a constant speed of 500 mi./hr. when the pilot ignites two rocket assist-units, each of which develops a forward thrust of 1000 lb. for 12 sec. If the velocity of the airplane is 530 mi./hr. at the end of the 12 sec., find the time average of the increase ΔR in air resistance. The weight of the rocket fuel is negligible compared with the weight of the airplane. <u>Ans</u>. $\Delta R = 178$ lb.

2. Determine the constant drawbar pull P required to increase the velocity of a 1500 ton train of freight cars from 30 mi./hr. to 50 mi./hr. up a 1 per cent grade in 3 min. Train resistance is 10 lb./ton. <u>Ans</u>. $P = 60{,}200$ lb.

3. A particle of mass m starts from rest and moves in a horizontal straight line under the action of a constant force P. Resistance to motion is proportional to the square of the velocity and is R = kv . Determine the total impulse I on the particle from the time it starts until it reaches its maximum velocity.
Ans. $I = m\sqrt{P/k}$

4. A 2 lb. body which has a velocity of 10 ft./sec. to the left is struck with an impact force F acting to the right on the body as represented in the graph. Approximate the loading by the dotted triangle and determine the final velocity v of the object.
Ans. v = 38.3 ft./sec. to the right

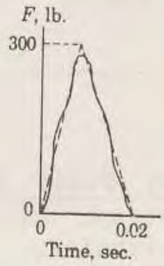


Fig.4

5. An 8000 lb. rocket sled is propelled by six rocket motors each with an impulse rating of 30,000 lb.sec. The rockets are fired at $\frac{1}{4}$ sec. intervals starting from rest, and the duration of each rocket is 1.5 sec. If the velocity of the sled is 600 ft./sec. in 3 sec. from the start, compute the time average R of the total resistance to motion. Neglect the loss of weight due to exhaust gases compared with the total weight of the sled. Ans. R = 10,330 lb.

6. Careful measurements made during the impact of the metal cylinder against the spring-supported plate disclose a semielliptical relation between the contact force F and the time t as shown. Determine the rebound velocity v of the cylinder if it weighs 8 oz. and strikes the plate with an initial velocity of 20 ft./sec.
Ans. v = 11.0 ft./sec.

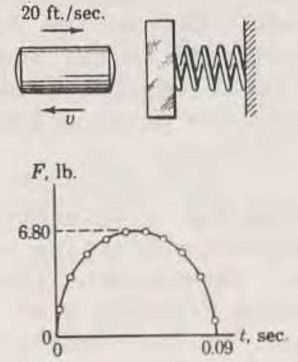


Fig.6

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16. **Diversion of Steady Fluid Streams:** The principle of impulse and momentum finds important application in the analysis of the dynamics of fluids. In particular, consider the diversion of a steady stream of fluid by a fixed section of pipe or a vane as represented in Fig.10. The fluid enters with a velocity V_1 through a cross-sectional area A_1 normal to the flow and leaves with a velocity V through a cross-sectional area A_2 normal to the flow. With steady flow there is no accumulation or depletion of fluid in the system. The free-body diagram of the section of fixed pipe and the fluid within it is shown in part (a) of the figure where ΣF is the resultant of all external forces acting on the system isolated. This resultant will include the weight of the pipe and fluid, if appreciable, and all external supporting forces applied to the pipe section. Also included are the forces due to the static pressure (gauge) across the entrance and exit sections exerted on the fluid within the system by the fluid external to the system. For fluid flow across an open vane the pressure in the stream is atmospheric, in which case there is no force due to static pressure which contributes to ΣF.

The force ΣF may be determined by a direct application of Eq.(23) which states the resultant force on any given mass system equals the time rate of change of the momentum of the system. Since there is no change in momentum of the pipe, the time rate of change of momentum is the vector difference between the rate at which momentum leaves the system and the rate at which momentum enters the system. If m' stands for the mass rate of flow, Eq.(23) gives:

$$\Sigma F = m' \Delta v \qquad \text{.................. (24)}$$

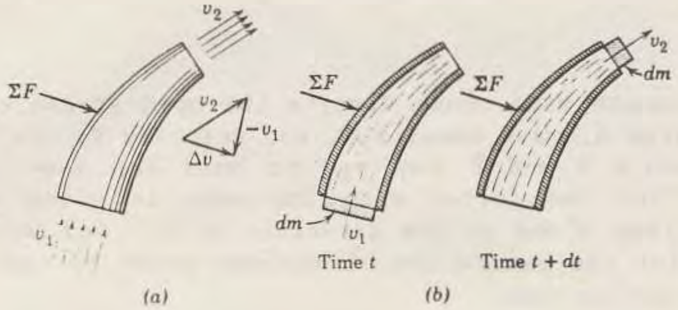


Fig.10

where $\Delta v = v_2 \longrightarrow v_1$. Since this is a vector equation, ΣF will have the same direction as Δv. Equation(24) may be applied directly as a vector equation, or it may be used in terms of its x- and y-components.

The impulse-momentum equation for steady fluid flow may also be obtained by an incremental analysis. Figure 10b. illustrates the system at time t and also at time t + dt. The system consists of the fixed pipe, the fluid within it, and the fluid of mass dm which will enter the pipe in time dt. After the interval dt the same system occupies the position shown in the right-hand view of Fig.10b., where a mass of fluid equal to dm. has emerged from the section. The momentum of the fluid within the pipe between the two ends at time t is identical with that between the same two ends at time t + dt, and therefore the difference between the momental of the system at the two times is $v_2\ dm \longrightarrow v_1\ dm$ or $\Delta v.dm$. Since this change occurs in time dt, the time rate of change of momentum is $\Delta v \frac{dm}{dt}$ or $m' \Delta v$ which equals the resultant force ΣF on the system as given by Eq.(24).

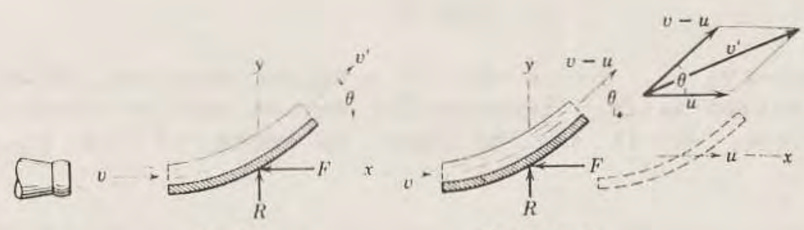
When the entrance and exit areas for the fluid flow are not the same, it is necessary to apply the equation of continuity which merely accounts for the constancy of mass flow. If ρ is the mass density, then $\rho_1 A_1 V_1 = \rho_2 A_2 V_2$. If the fluid is a liquid, the density will be essentially constant.

Equation(24) is independent of the properties of the fluid and may be used to analyze any type of steady mass flow whether it be liquid, gas, or particles of mass such as sand.

One of the most important applications of the momentum equation for fluid flow occurs when the diverting body is in motion, such as a pump vane or turbine blade. The sample problem which follows will illustrate this condition.

SAMPLE PROBLEM

1. The smooth vane shown diverts the open stream of fluid of cross-sectional area A, mass density ρ, and velocity V. (a) Determine the force components R and F required to hold the vane in a fixed position. (b) Find the forces when the vane is given a constant velocity μ less then V and in the direction of V. (c) Determine the optimum speed μ for the generation of maximum power by the action of the fluid on the moving vane.



Fixed vane   Moving vane

Solution: Part(a). The free-body diagram of the vane together with the fluid portion undergoing the momentum change is shown. The momentum equation may be applied to the isolated system for the change in motion in both the x- and y- directions. With the vane stationary the magnitude of the exit velocity V' equals that of the entering velocity V with fluid friction neglected. The changes in the velocity components are, then:

$$\Delta V_x = v' \cos \theta - v = -v(1-\cos \theta) \text{ and } \Delta v_y = v \sin \theta - 0 = v \sin \theta.$$

The mass rate of flow is $m' = \rho Av$, and substitution into Eq.(24) gives:

$[\Sigma F_x = m \Delta v_x]$ $\quad -F = \rho Av [-v(1-\cos \theta)], \quad F = \rho Av^2 (1-\cos \theta),$ Ans.

$[\Sigma F_y = m \Delta v_y]$ $\quad = \rho Av [v \sin \theta], \quad R = \rho Av^2 \sin \theta$ Ans.

Part(b). In the case of the moving vane the final velocity of the fluid upon exit is the vector sum of the velocity μ of the vane plus the velocity of the fluid relative to the vane. This combination is shown in the velocity diagram to the right of the figure for the exit conditions. The relative velocity is that which would be measured by an observer moving with the vane. This observer would measure v-μ feet of fluid passing over the vane per second, and the direction of this relative velocity is tangent to the vane at exit. The combination of these two velocity components gives the final absolute fluid velocity V' as shown. The x-component of V' is the sum of the components of its two parts, so $V'_x = (v-\mu) \cos \theta + \mu$. The change in x-velocity of the stream is:

The y-component of V' is $(v-\mu)\sin\theta$, so that the change in the y-velocity of the stream is $\Delta V_y = (v-\mu)\sin\theta$.

The mass rate of flow m' is the mass undergoing momentum change per unit of time. This rate is the mass flowing over the vane per unit time and not the rate of issuance from the nozzle. Thus:

$$m' = \rho A\,(v-\mu)$$

The impulses-momentum principle applied to the positive coordinate direction gives:

$$[\Sigma F_x = m'\,\Delta v_x] \qquad -F = \rho A(v-\mu)\,[-(v-\mu)\,(1-\cos\theta)],$$

$$F = \rho A(v-\mu)^2\,(1-\cos\theta) \qquad \underline{\text{Ans.}}$$

$$[\Delta F_y = m'\,\Delta v_y] \qquad R = \rho A(v-\mu)^2\,\sin\theta \qquad \underline{\text{Ans.}}$$

Part(c). Since R is normal to the velocity of the vane, it does not work. The work done by the force F shown is negative, but the power developed by the force (equal and opposite to F) exerted by the fluid on the moving vane is:

$$[P = F\mu] \qquad P = \rho A(v-\mu)^2\,\mu\,(1-\cos\theta).$$

The velocity of the vane for maximum power for the one blade in the stream is specified by:

$$\left[\frac{dP}{du} = 0\right] \qquad \rho A(1-\cos\theta)\,(v^2 - 4\mu v + 3\mu^2) = 0$$

$$(v - 3\;)\,(v - \mu) = 0m \;\; \mu = \frac{v}{3} \qquad \underline{\text{Ans.}}$$

The second solution $\mu = v$ gives a minimum condition of zero power. An angle $\theta = 180$ deg. completely reverses the direction of the fluid and clearly produces both maximum force and maximum power for any value of μ.

PROBLEMS

1. When the air only of a sand-blasting gun is turned on, the force of the air on a flat surface normal to the stream and 6 in. from the nozzle is 2 lb. With the nozzle in the same position the force increases to 5 lb. when the sand is admitted to the stream. If sand is used at the rate of 10 lb./min., find the velocity V of the particles of sand as they strike the surface. <u>Ans</u>. V = 580 ft./sec.

2. The tender of a locomotive traveling at the constant speed of 60 mi./hr. replenishes its supply of water by scooping it up from a trough between the rails at the rate of 6 ft^3/sec. Find the added resistance R to motion due to the action of the scoop.
<u>Ans</u>. R = 1024 lb.

3. The 180 deg. pipe return discharges salt water with a density of 64.4 lb./ft^3. into the atmosphere at a constant rate of 3 ft^3/sec. The static pressure in the water at section A is 7 lb./in^2 above atmospheric pressure. The flow area of the pipe at A is 24 in^2. and that at discharge is 11.8 in^2. If each of the six flange-bolts is tightened with a torque wrench so that it is under a tension of 2000 lb., determine the average pressure p. on the gasket between the two flanges. The flange area in contact with the gasket is 10.4 in^2.
<u>Ans</u>. p = 67.7 lb./in^2.

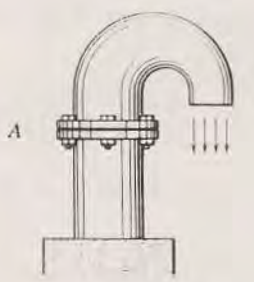


Fig. 3

Fig. 4

4. A ball of weight W is supported in a vertical jet of water with weight density μ . If the stream of water issuing from the nozzle has a diameter d and velocity μ, determine the height h above the nozzle at which the ball is supported. Assume that the jet remains intact and that there is no energy loss in the jet.
<u>Ans</u>. $h = \frac{1}{2y} \left(\mu^2 - \left[\frac{4Wg}{\pi\mu u d^2}\right]^2\right)$

5. A stream of water flowing at the rate of 500 gal./min. from a 2 in diameter nozzle is diverted by the fixed vane. The vane is positioned so that $\frac{2}{3}$ of the stream is diverted up and the remainder is deflected down and back as shown. Determine the total force F required to hold the vane in place. (10 lb. of water per 1 imperial gallon).

6. Freshwater is supplied to the fixed pipe under a static pressure fo 50 $lb./in^2$. and issues from the nozzles A and B at velocities of 70 and 75 ft./sec., respectively. The inside pipe diameters at A, B, & C in that order are 1 in., 2 in., & 4 in. Calculate the average tension T and horizontal shear Q. in the pipe at section C. Neglect the small-weight of the pipe and water within it.
Ans T = 682 lb., Q = 201 lb.

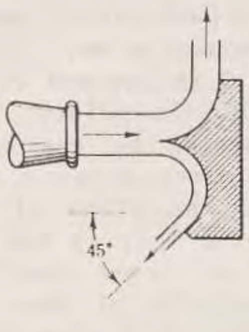


Fig. 5

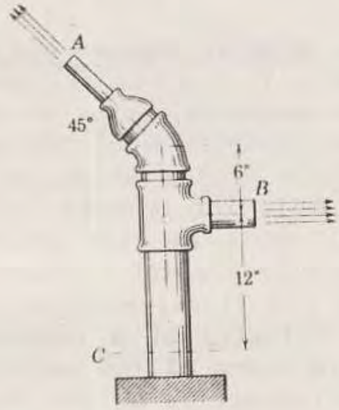


Fig. 6

7. The experimental aircralf has a total weight of 10,000 lb. and is designed for VTOL (vertical take-off and landing). Movable wing sections divert the engine jet-stream downward to provide the component of vertical lift. Each of the two engines sucks in 83 lb. of air per second at a density of 0.076 $lb./ft^3$. and discharges the heated air with a velocity of 2000 ft./sec. The mass of fuel added to the exhaust stream is small and may be neglected. For the given conditions determine the angle θ which will initiate vertical take off, and find the inlet area A for each jet engine if the horizontal thrust on the airplane is zero. Ans. $\theta = 14°5'$, $A = 2.24\ ft^2$.

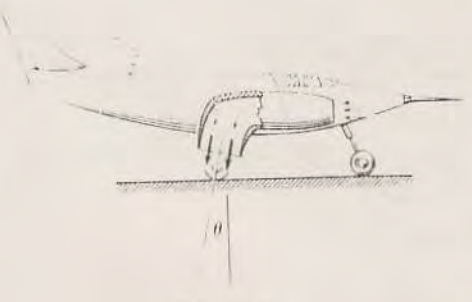


Fig. 7

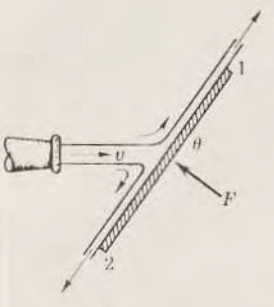


Fig. 8

8. A jet of fluid with velocity V, cross-sectional area A, and mass density ρ impinges on a fixed slanted trough shown in section. Some of the fluid is diverted in each of the two directions. If the trough is smooth, the velocity of both diverted streams remains V, and the only force which can be exerted on the fluid is normal to the surface of the trough. By writing the impulse-momentum equations for the directions along and normal to the trough determine the force F required to support the trough and the rate of flow Q (volume per unit time) in each of the two directions.

<u>Ans</u> $F = \rho A v^2 \sin \theta$;

$$Q_1 = \frac{Q}{2}(1 + \cos\theta),\ Q_2 = \frac{Q}{2}(1 - \cos\theta).$$

---

17. **<u>Angular Impulse and Momentum</u>:** A particle of mass m moving with velocity V, Fig.(11 a), has a linear momentum mv. The moment of this linear-momentum vector about the point O is mvr and is defined as the ANGULAR MOMENTUM H of the particle about the point. The angular momentum is a vector which may be represented by the conventional right-hand rule for moments. The angular momentum of a rigid body, Fig.(11 b), about an axis perpendicular to the plane of motion and passing through a point O fixed in the moving body is the sum of the moments of the linear momenta of all its particles about the axis. Since the velocity of a representative particle of mass m_i may be expressed in terms of the velocity components of O plus the velocity of m_i with respect to O, the linear momentum of m will be the vector sum of the components shown in the figure. Thus the angular momentum of the particle about O will be the sum of the moments of these linear momentum components about O. Summing up these terms for all particles gives for the total angular momentum of the body about point O fixed in the body.

$$H_o = \Sigma m_i r^2 w + \Sigma m_i V_{o_y} X - \Sigma m_i V_{o_x} Y,$$

where the counterclockwise direction is arbitrarily taken as positive.

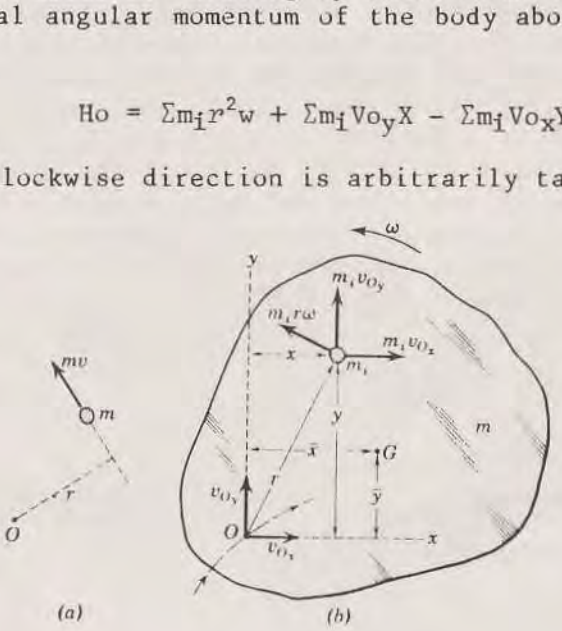


Fig. 11

Introducing the definition of the moment of inertia about O and the coordinates to the mass center gives:

$$Ho = IoW + \bar{x}\ m\ Vo_y - \bar{y}\ m\ Vo_x \ \ldots\ldots\ldots\ldots\ (25)$$

It should be observed that for clockwise rotation the signs of the last two terms in this equation would be reversed if Ho is measured positive clockwise. Equation(25) finds it greatest use when the axis through O is fixed (pure rotation), which gives:

$$Ho = IoW$$

and also when the mass center G is used as the reference point, in which case:

$$H = \bar{I}W$$

The linear momentum of a body is the vector G = mv which has the direction of the velocity of the center of mass. Since angular momentum equals the moment of linear momentum, the position of the vector mv may be located for each of the three types of plane motion shown in Fig.(12). In part (a) of the figure for translation the momentum vector mv passes through G. This condition is easily seen since the resultant moment of the linear momentum of all particles about G is:

$$\Sigma m_i \bar{v} y = \bar{v}\Sigma m_i y = \bar{v} m \bar{y} = 0,$$

where m is the mass of a representative particle. It follows that the angular momentum of a translating body about an axis through a moving or fixed point such as A is H = mvb.

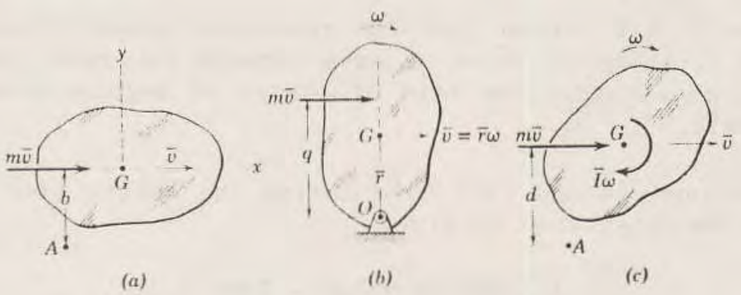


Fig.12

For rotation about a fixed axis at O in Fig.(12 b) the moment of the linear momentum equals the angular momentum, so that:

$$m\bar{v}q = IoW, \text{ and } q = \frac{IoW}{m\bar{v}} = \frac{ko^2}{r}.$$

Thus the linear-momentum vector passes through the center of percussion relative to O. If the body rotates about a fixed axis through G, the linear momentum is zero, and the angular momentum is IW, a free vector normal to the plane of rotation with all the properties of a couple. In this event the angular momentum is the same about all parallel axes, fixed or moving.

In the case of general plane motion, Fig.(12 c), if the angular momentum of the body about the mass center G is represented by IW, which has the properties of a couple or free vector, the linear momentum mv must pass through G. The angular momentum of the body about some other axis, fixed or moving, through a point A is H_A = mvd + IW. It may be noted that mv and IW may be replaced by a single vector mv with a different but parallel line of action if desired. This combination is identical to that for a force and a couple.

The relation between the angular momentum of a body and the applied moments is obtained from the rotational equation of motion. For rotation about the fixed axis through O, Fig.(12 b), the sum of the moments of all forces about O is $\Sigma Mo = Io\alpha = Io(\frac{dw}{dt})$. Since the moment of inertia Io of the rigid body is constant, the motion equation may be written:

$$\Sigma\ Mo = \frac{d}{dt}\,(IoW) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (26)$$

In precisely the same manner the moment equation about the mass center for the case of any plane motion may be written:

$$\Sigma\bar{M} = \frac{d}{dt}\,(\bar{I}w) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (27)$$

In words Eqs.26 & 27 state that the resultant moment about the fixed axis of pure rotation or about an axis through the mass center in any plane motion equals the time rate of change of angular momentum about respective axis.

Equations(26) and (27) hold during the entire time of motion, and each may be integrated to give:

$$\int_o^t \Sigma Mo.dt = IoW - IoWo \quad \ldots\ldots\ldots\ldots\ldots \quad (28)$$

for rotation about the fixed axis through O and

$$\int_o^t \Sigma\bar{M}\ dt = \bar{I}W - IWo \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (29)$$

for a reference axis through the mass center in plane motion. In each case the angular velocity changes from Wo at time t = 0 to W at time t. These equations state that "the total angular impulse equals the corresponding net change in angular momentum.

The angular impulse-momentum equations are analogous to the linear impulse-momentum equations developed in article 15. Comparison shows that the dimensions of angular impulse and angular momentum are [moment] x [time], (lb.ft.sec.), whereas those for linear impulse and linear momentum are [force] x [time], (lb.sec.). Thus these quantities can not be added.

The rolling wheel, Fig.(13), is an important special case of plane motion which deserves separate comment. It was shown that the equation $\Sigma Mc = Ic\,\alpha$ holds with respect to the instant center C of zero velocity at all times during the motion provided the wheel does not slip and provided the geometric center of the wheel is also the mass center. This motion equation may be written: $\Sigma Mc = \frac{d}{dt}(IcW)$ or integrated to give: $\int_o^t \Sigma Mc.dt = IcW - IcWo$. Integration is permitted since the same differential relation holds throughout rolling under the conditions stated. In this case the moment axis is not attached to the wheel but moves with it and always coincides with the instantaneous center C. The advantage of this equation for a rolling wheel is that the contact forces at C are eliminated from the equation. Considerable caution is warranted, however, since the expression must not be used except for a symmetrical wheel which does not slip.

Equations (126) and (27) were developed for rigid body motion, but it is important to recognize more general applicability to any system of particles of constant mass. In Fig.(14) let m_i be the mass of a representative particle of any system, and consider

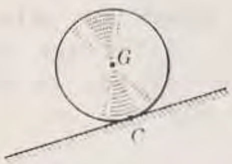


Fig. 13

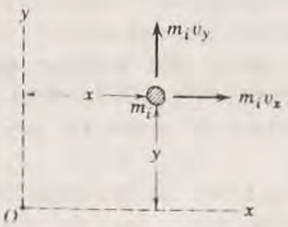


Fig. 14

the components of its motion in the x-y plane. From the principle of linear impulse and momentum the resultant force on m_i in the x-direction and that in the y-direction are $Fx + fx = \frac{d}{dt}(m_i\, v_x)$; $Fy + fy = \frac{d}{dt}(m_i\, V_y)$, where Fx and Fy stand for forces applied from sources external to the system and f_x and f_y represent forces applied from sources internal to the system. The resultant moment in a counter-clockwise sense about a fixed point O is:

$$M_i = (F_y + f_y)\,x - (F_x + f_x)\,y = x\,\frac{d}{dt}(m_i\,V_y) - Y\,\frac{d}{dt}(m_i\,V_x).$$

Differentiation will show that this expression is the same as:

$$M_i = \frac{d}{dt}(xm_i v_y) - \frac{d}{dt}(ym_i v_x),$$

or $$M_i = \frac{d}{dt}(xm_i v_y - ym_i v_x) = \frac{dHi}{dt}$$

Thus the resultant M_i about 0 of all forces on m equals the time rate of change of angular momentum about 0. By adding all such equations written for every particle of the system there results:

$$\Sigma M_i = \Sigma \frac{dHi}{di} = \frac{d}{dt} \Sigma H_i$$

The contribution to ΣM_i by the internal forces is zero since they occur in pairs of equal and opposite forces and their moments cancel. If the resultant moment of external forces about 0 is denoted by ΣMo and the sum ΣH_i of the angular momenta of all particles about 0 by Ho, the impulse-momentum equation becomes:

$$\Sigma Mo = \frac{dHo}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (30)$$

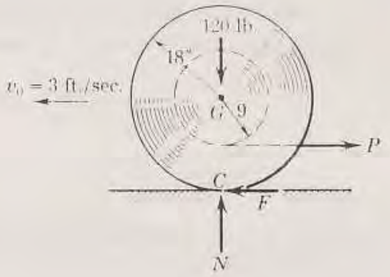
Similar analyses in the two other coordinate planes will disclose similar equations, and thus Eq.(30) may be considered the vector combination of these three relations. Consequently, it may be stated that the resultant vector moment about any fixed point for a system of particles of constant total mass equals the time rate of change of angular momentum of the system about an axis through the point parallel with the vector moment axis. Althrough the proof will not be given here, Eq.(30) also holds with respect to a moving axis through the center of mass of any such system.

The relation expressed by Eq.(30) applied either to a fixed point 0 or to the mass center G constitutes one of the most general of the derived relations of mechanics. It permits the writing of moment equations of motion where the angular momentum may be changing in direction as well as in magnitude. The case of change in direction of the angular momentum is dealt with later on gyroscopic motion.

SAMPLES PROBLEMS

1. The force P which is applied to the cable wrapped around the central hub of the symmetrical wheel is increased slowly according to P = 1.5 t, where P is in pounds and t is in seconds. Determine the

angular velocity w of the wheel 10 sec. after P is applied if the wheel is rolling to the left with a linear velocity of its center of 3 ft./sec. at time t = 0. The wheel weighs 120 lb. with a centroidal radius of gyration of 10 in. and rolls without slipping.



Solution I. The free-body diag. of the wheel is shown. The correct direction of the friction force is established by the necessity for a positive clockwise moment of forces about G to produce the resulting clockwise angular acceleration. Direct application of the angular impulse-momentum equation with respect to the mass center, Eq.(29) gives:

$$\left[\int_0^t \Sigma\bar{M}dt = \Delta(\bar{I}w)\right]:$$

$$\int_0^{10} \left(\frac{18}{12} F - \frac{9}{12} \times 1.5t\right) dt = \frac{120}{32.2} \left(\frac{10}{12}\right)^2 \left[w - \left(-\frac{3}{18/12}\right)\right],$$ where the positive direction is taken as clockwise. The force F is a variable and so must be left under the integral sign. The second equation needed to eliminate F is that of linear impulse and momentum which applies to the motion of the center of mass of any system. Thus:

$$\left[\int_0^t \Sigma F.dt = \Delta\ (m\bar{v})\right] : \quad \int_0^{10} (1.5t - F)\ dt = \frac{120}{32.2}\left[\frac{18}{12} w - (-3)\right]$$

The integral involving F is easily eliminated between the two equations, and the result is:

w = 3.13 rad./sec. clockwise <u>Ans</u>.

Solution II. Since the wheel is symmetrical, an axis moving with the instant center C may be used. The necessity of a simultaneous solution is eliminated since F does not appear in the equation. Hence:

$$\left[\int_0^t \Sigma M_C dt = \Delta\ (I_C w)\right]:$$

$$\int_0^{10} \frac{9}{12} \times 1.5t.dt = \frac{120}{32.2}\left[\left(\frac{10}{12}\right)^2 + \left(\frac{18}{12}\right)^2\right]\left[w-\left(-\frac{3}{18/12}\right)\right],$$

which gives w = 3.13 rad./sec.

Solution by use of the instant center is not permitted if any slipping occurs or if the geometric center and center of gravity do not coincide.

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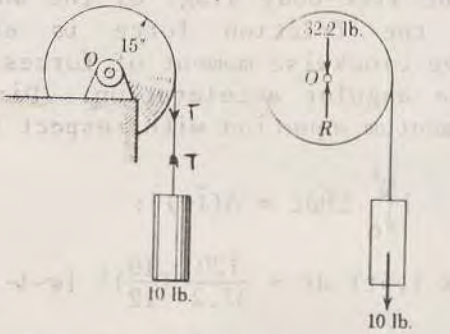
2. Determine the velocity v of the 10 lb. weight 4 sec. after it is released from rest. The drum weighs 32.2 lb. with a radius of gyration of 8 in. and has negligible friction in its bearing at O.



Solution: The drum and the weight may be isolated separately with the angular impulse-momentum equation applied to the drum and the linear impulse-momentum applied to the weight. The tension T in the cable may be eliminated and the resulting expression solved for v. A simpler method involves the application of Eq.(30) in integral form to a fixed axis for both parts considered a single system. The fixed axis will be taken at O to eliminate the 32.2 lb. weight and the bearing reaction R from the equation. From the free-body diagram of the entire system the resultant moment about O of all external forces is that due to the weight only. Also the angular momentum of the weight about O is the moment of its linear momentum. Thus:

$$[\Sigma \text{ Mot} = \Delta \text{ Ho}] \quad 10 \times \frac{15}{12} \times 4 = \frac{32.2}{32.2}\left(\frac{8}{12}\right)^2 \frac{v}{15/12} + \left(\frac{10}{32.2}v\right) \times \frac{15}{12}$$

$$v = 67.2 \text{ ft./sec.} \qquad \underline{\text{Ans.}}$$

PROBLEMS

1. The rotor of a steam turbine weighs 80 lb. with a radius of gyration of 12 in. and requires 6 min. to come to rest from a speed of 10,000 rev./min. after the steam is shut off. Determine the average value of the resisting moment M due to internal friction.
Ans. M = 7.23 lb.ft.

2. The center of the homogeneous solid cylinder is given an initial velocity of 2 ft./sec. up the incline. Determine the time t required for it to reach a velocity of 4 ft./sec. down the incline if it rolls without slipping. Ans. t = 2.81 sec.

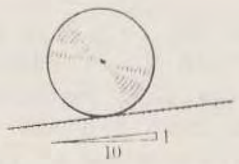


Fig.2

3. A 3 oz. bullet is travelling at a speed of 2000 ft./sec. in the plane of rotation of the uniform bar weighs 10 lb. If the bar is swinging and the bullet strikes it when in the vertical position, determine the angular velocity w of the bar just before collision so that the angular momentum of the system about O is zero at this instant. Ans. w = 31.3 rad./sec.

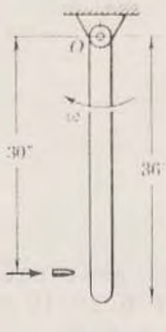


Fig.3

4. Determine the torque M which must be applied to the deflector at O in order to support the 90 deg. diversion of an open stream of freshwater flowing in a horizontal plane with a velocity v of 100 ft./sec. and at the rate of 4 lb./sec. Ans. M = 2.07 lb.ft.

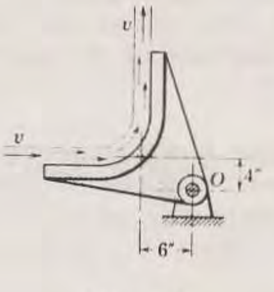


Fig. 4

5. The pulley weighs 30 lb., has a radius of gyration of 10 in., and carries the 45 lb. load W. Constant tensions of 50 lb. and 40 lb. are applied to the vertical hoisting cables shown. If the velocity V of W is 6 ft./sec. down and the angular velocity W of the pulley is 8 rad./sec. counterclockwise at time t = 0, determine V and W after the cable tensions have been applied for 4 sec. Note the independence of the results.

Ans. V = 19.8 ft./sec. up; w = 69.3 rad./sec. clockwise

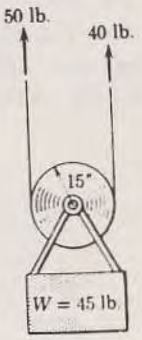


Fig. 5

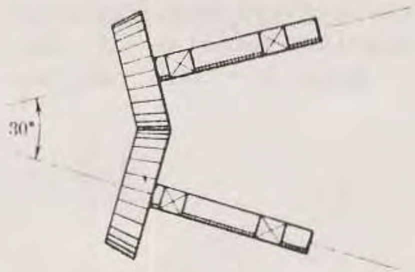


Fig. 6

6. Each identical bevel gear and attached shaft has a moment of inertia about its own axis of 0.20 lb.ft.sec.^{2} and rotates in fixed bearings at a speed of 1000 rev./min. while in mesh. Determine the magnitude of the total angular momentum H of the system.

Ans. H = 10.84 lb.ft.sec.

7. In the centrifugal-pump impeller shown water flows to the straight radial vanes in an axial direction and leaves the vanes with a velocity V whose tangential component is the rim speed of the impeller. If the pump handles 2000 gal./min. at a speed of 840 rev./min., what is theoretical torque M on the impeller shaft-required to impart the angular momentum to the water? If the actual power required to run the pump is 30 h.p., what is the efficiency e of the pump?

Ans. M = 132 lb.ft.; e = 70.4%

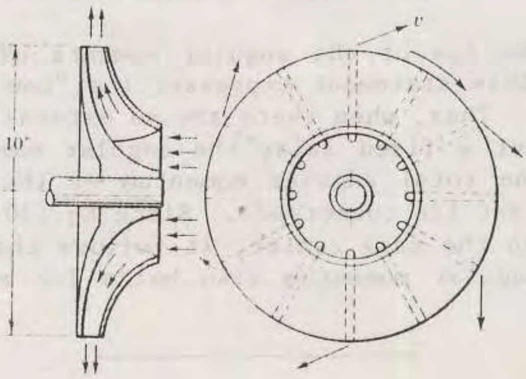


Fig.7

18. **Conservation of Momentum:** The principle of linear impulse and momentum for any mass system is expressed by Eqs.(22) and states that the resultant external force in any direction on the system equals the time rate of change of the linear momentum of the system in that direction. If the resultant force in, say, the x-direction is zero during any interval of time, it follows that the time rate of change of the momentum in that direction is also zero. Hence:

$$G_x = \Sigma \, mv_x = \text{constant}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (31)$$

where $\Sigma \, mv_x$ is the sum of the linear momenta of the several parts of the system in the x-direction. This statement expresses the "Law of Conservation of Linear Momentum". Thus the linear momentum in any direction for a system of bodies remains constant (is conserved) as long as there is no resultant external force on the system in that direction. The principle finds particular use in describing the interactions of bodies such as the recoil of a gun or the collision of two objects.

The principle of angular impulse and momentum about a fixed axis for any system of bodies is expressed by Eq.(30) and states that the resultant moment about a fixed axis O. equals the time rate of change of angular momentum of the system about that axis. If the resultant moment about any such axis is zero during an interval of time, it follows that the time rate of change of angular momentum about that axis is also zero. Hence:

$$Ho = \Sigma I o w = \text{constant} \quad \ldots\ldots\ldots\ldots\ldots\ldots (32)$$

where $\Sigma I o w$ is the sum of the angular momenta of all parts of the system about O. This statement expresses the "Law of Conservation of Angular Momentum". Thus, when there are no externally applied moments on any system about a fixed axis, the angular momentum of each part may change, but the total angular momentum of the system about this axis remains constant (is conserved). Since Eq.(30) also applies to a moving axis through the mass center, it follows that the principle of conservation of angular momentum also holds for a moving centroidal axis.

SAMPLE PROBLEMS

1. The 2 oz. bullet traveling at 2000 ft./sec. strikes the 10 lb. block centrally and is embedded within it. If the block is sliding on a smooth horizontal plane with a velocity of 40 ft./sec. in the direction shown just before impact, determine the velocity V. of the block and bullet and its direction θ immediately after impact.

Solution: Since the force of impact is internal to the system composed of the block and bullet and since there are no other external forces acting on the system, it follows that the linear momentum of the system is conserved in both the x- and y- directions. Thus:

$$[\Delta G_x = 0] : \quad (10 \times 40 \cos 30°) + 0 = (10 + \frac{2}{16})\, v_x,$$

$$v_x = 34.2 \text{ ft./sec.}$$

$$[\Delta G_y = 0) : \quad (10 \times 40 \sin 30°) + \frac{2}{16}.2000 = (10 + \frac{2}{16})v_y,$$

$$v_y = 44.4 \text{ ft./sec.}$$

The acceleration of gravity g has been omitted since it appears in each term and cancels. The final velocity is given by:

$$[v = \sqrt{v_x^2 + v_y^2}] \quad : \quad v = \sqrt{(34.2)^2 + (44.4)^2} = 56.1 \text{ ft./sec.} \quad \underline{\text{Ans}}.$$

The direction of the final velocity is given by:

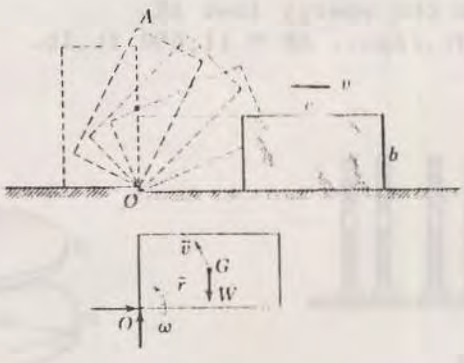
$$[\tan \theta = \frac{v_y}{v_x}] \quad : \quad \tan \theta = \frac{44.4}{34.2} = 1.30 \quad \ldots\ldots\ldots\ldots\ldots \quad \underline{\text{Ans}}.$$

---

2. The uniform rectangular block of dimensions shown is sliding to the left on the horizontal surface with a velocity v when it strikes the small step in the surface. Assume negligible rebound at the step, and compute the minimum value of v which will permit the block to pivot about the edge of the step and just reach the standing position shown with no velocity. Compute the emergy loss ΔE for $b = c$.

Solution: It will be assumed that the edge of the step O acts as a latch on the corner of the block, so that the block pivots about O. Furthermore, the height of the step is assumed negligible compared with the dimensions of the block. During impact the only force which exerts a moment about O is the weight W, but the angular impulse due to the weight is extremely small since the time of impact is negligible. Thus angular momentum about O may be said to be conserved.

The initial angular momentum of the block about O just before impact is the moment of its linear momentum and is $Ho = (\frac{W}{g})\ v\ (\frac{b}{2})$. The velocity of the center of mass G immediately after impact is $\bar{v}$,



and the angular velocity is $w = \frac{\bar{v}}{\bar{r}}$. The angular momentum about O just after impact when the block is starting its rotation about O is:

$$[Ho = I_o w] \ : \ Ho = \left[\frac{1}{12}\frac{W}{g}(b^2+c^2)\ \frac{W}{g}\left(\left[\frac{c}{2}\right]^2 + \left[\frac{b}{2}\right]^2\right)\right] w = \frac{W}{3g}(b^2+c^2)w.$$

Conservation of angular momentum gives:

$$[\Delta Ho = 0] : \frac{W}{3g}(b^2 + c^2)w = \frac{W}{g} v \frac{b}{2}, \quad w = \frac{3vb}{2(b^2 + c^2)}.$$

This angular velocity will be sufficient to raise the block just past position A if the kinetic energy of rotation equals the increase in potential energy. Thus:

$$[\Delta T + \Delta V_g = 0] : \frac{1}{2} Iow^2 - W\left(\sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2} - \frac{b}{2}\right) = 0,$$

$$\frac{1}{2}\frac{W}{3g}(b^2 + c^2)\left[\frac{3vb}{2(b^2 + c^2)}\right]^2 - \frac{W}{2}\left(\sqrt{b^2 + c^2} - b\right) = 0$$

$$v = 2\sqrt{\frac{g}{3}\left(1 + \frac{c^2}{b^2}\right)\left(\sqrt{b^2 + c^2} - b\right)} \ \ldots\ldots\ldots\ldots \ \underline{\text{Ans}}.$$

The percentage loss of energy is:

$$\frac{\Delta E}{E} = \frac{\frac{1}{2} mv^2 - \frac{1}{2} I_o w^2}{\frac{1}{2} mv^2} = 1 - \frac{ko^2 w^2}{v^2} = 1 - \left(\frac{b^2 + c^2}{3}\right)\left[\frac{3b}{2(b^2 + c^2)}\right]^2.$$

$$= 1 - \frac{3}{4\left(1 + \frac{c^2}{b^2}\right)}, \quad \frac{\Delta E}{E} = 62.5 \text{ per cent for } b = c \ \ldots\ldots\ldots\ldots \ \underline{\text{Ans}}.$$

PROBLEMS

1. The 4 oz. projectile is fired with a velocity of 2000 ft./sec. and picks up the four washers, each of which weighs 3 oz. Find the common velocity v of projectile and washers following the interaction. Determine the energy loss ΔE.

Ans. $v = 500$ ft./sec., $\Delta E = 11{,}650$ ft.lb.

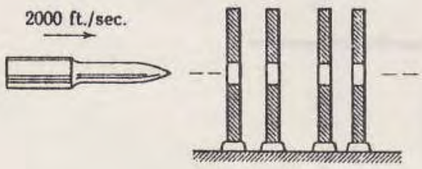


Fig. 1

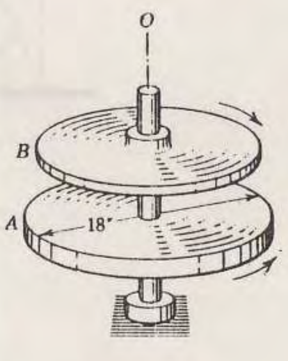


Fig. 2

2. The uniform 20 lb. circular disk A is spinning freely at 400 rev./min. about the vertical shaft. Disk B weighs 10 lb. with a radius of gyration of 4 in. If disk B is spinning at 200 rev./min. in the opposite direction as it is allowed to drop onto disk A, determine the final common velocity N of the two disks after slipping ceases.

Ans. N = 301 rev./min.

3. The barrel and breech of a 3 in. anti aircraft gun weigh 1800 lb. The projectile weighs 15 lb. and has a muzzle velocity of 3000 ft./sec. The recoil of the gun is checked by a combination of springs and oil dampers so that the force F of the recoil mechanism on the moving barrel is essentially constant. Determine F if the recoil distance is 16 in. Neglect the weight of the gases and assume that the gun acquires its full recoil velocity before the recoil mechanism begins to act.

Ans. F = 13,100 lb.

4. The third and fourth stages of a rocket are traveling at 5000 mi./hr. when the third stage runs out of fuel and its thrust drops to zero. An instant after this occurs the fourth stage ignites, and its thrust against the third stage causes separation with no other forces of interaction between the two parts. At this condition the empty third-stage case has a mass of 4 $lb.ft^{-1}$ $sec.^{2}$, and the fourth stage has a mass of 2 $lb.ft^{-1}.sec.^{2}$. If the relative velocity of separation is 40 ft.sec. and separation occurs ¼ sec. after the fourth stage ignites, determine the velocity v. of the fourth stage as it leaves the third stage and the average thrust T of the fourth stage during separation.

Ans. v = 5018 mi./hr., T = 213 lb.

5. A ballistic pendulum consists of a 100 lb. box of sand suspended by a wire as shown. A 2 oz. bullet traveling horizontally is embedded in the sand, and the pendulum is observed to swing through an angle θ = 10 deg. Determine the initial velocity v of the bullet. What percentage e of the energy of the bullet is lost from the system?

Ans. v = 3070 ft./sec., e = 99.88%

Fig. 5

Fig. 6

6. An earth satellite encircles the earth in the elliptical orbit shown. (a) By using polar coordinates r, θ with origin at the center of the earth show that the equation of motion in the θ- direction expresses the conservation of angular momentum for the satellite. (b) At the apogee A (farthest point from the earth) the satellite is 1200 mi. from the earth's surface, and at the perigee (closest point to the earth) it is 200 mi. above the earth's surface. The radius of the earth is 3960 mi. Under these conditions the velocity of the satellite at P is 18,170 mi./hr. What is the velocity V at A?

7. A uniform pole of length L is dropped at an angle θ with the vertical, and both ends have a velocity V as end A hits the ground. If end A pivots about its contact point during the remainder of its motion, determine the velocity v' with which end B hits the ground.

Ans. $v' = \sqrt{(9v^2/4)\sin^2\theta + 3gL\cos\theta}$

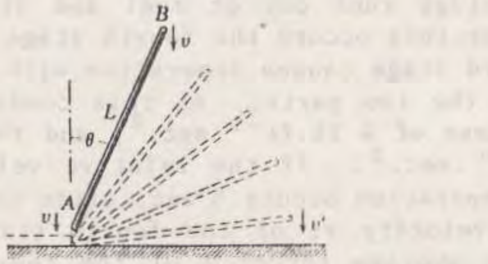


Fig. 7

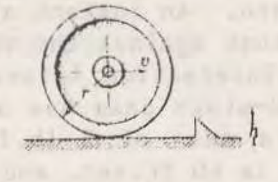


Fig. 8

8. Determine the minimum velocity v which the wheel may have and just roll over the obstruction. The centroidal radius of gyration of the wheel is k, and it is assumed that the wheel does not slip.

Ans. $v = \dfrac{r}{k^2 + r^2 - rh}\sqrt{2gh(k^2 + r^2)}$.

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19. Impact: The collision between two bodies where relatively large contact forces exist during a very short interval of time is called "IMPACT". Experimental verification of impact theory is difficult by reason of the extremely short time during which the contact forces act. With the advent of modern instrumentation, however, reliable data for the description of impact phenomena have become available.

As an introduction to impact consider the collinear motion of two spheres of masses m_1 and m_2, Fig. 15a, traveling with velocities v_1 and v_2. If v_2 is greater than v_1, collision occurs, and a short period of deformation takes place, Fig. 15b, until the contact area between the spheres ceases to increase. After this deformation a period of restoration takes place, and the spheres regain their original shape if the blow is not too severe or else retain a deformed shape if the impact is more severe. The spheres then continue to move with final velocities v_1' and v_2' as in Fig. 15c. Inasmuch as the contact forces are equal and opposite, the linear momentum of the system remains unchanged. Thus the law of conservation of momentum applies and:

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

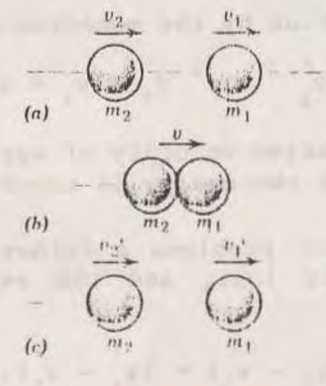


Fig. 15

All velocities are arbitrarily assumed positive to the right so that a negative sign will describe a velocity to the left.

In addition to the conservation of momentum, the energy of the colliding masses must be accounted for. The initial kinetic energy of the system before impact is divided into three parts after impact. First, some of the energy is retained in the form of kinetic energy on account of the rebound velocities of the sphere as a whole. Second, some of the initial energy is lost as a result of the generation of heat if the spheres are permanently deformed and by the generation of sound waves. Third, the impact forces cause internal vibrations of the spheres, and the resulting propagation and rebound of elastic waves within the spheres consume some of the initial energy. This third part of the total energy is usually difficult to account for and is by no means negligible in many impact-problems involving bodies whose shapes are other than spherical.

The classical theory of impact as presented in most treatment of mechanics neglects the internal energy of vibration. With this neglect and for the case of perfectly elastic impact the final kinetic energy must equal the initial kinetic energy.

Thus: $$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2,$$

or: $$m_1 (v_1 + v_1') (v_1 - v_1') = m_2 (v_2 + v_2') (v_2' - v_2).$$

The equation for the conservation of momentum may be written:

$$m_1 (v_1 - v_1') = m_2 (v_2' - v_2).$$

Dividing the energy equation by the momentum equation gives:

$$v_1 + v_1' = v_2 + v_2' \quad \text{or} \quad v_2 - v_1 = v_1' - v_2',$$

which shows that the relative velocity of approach equals the relative velocity of separation if the energy is conserved.

In most impact problems a rather large percentage of the energy of the system is lost, and the equation for the velocity difference is written:

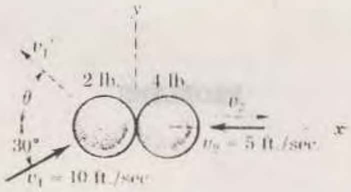
$$e (v_2 - v_1) = (v_1' - v_2').$$

The factor e, which may vary between zero and unity, is known as the "Coefficient of Restitution" and equals the ratio of the relative velocity of separation to the relative velocity of approach. If the impact occurs obliquely, only the components of the velocities in the direction of the force of impact should be used with the coefficient of restitution. In the classical theory of impact a coefficient of restitution of unity means "Elastic Impact" with no energy loss, and a coefficient of restitution of zero means an "Inelastic or Plastic Impact", where the bodies cling together after collision and the energy loss is a maximum.

Experimental determination of coefficients of restitution for spheres of various materials indicates that e varies greatly with the impact velocity but approaches unity when this velocity approaches zero. This condition is explained on the basis that the energy loss due to permanent deformation becomes less as the impact velocity decreases. Experiment also has shown that for given materials and for a given impact velocity the coefficient of restitution changes appreciably with the size and shape of the colliding bodies. This effect is due largely to the corresponding change in induced internal energy of vibration.

SAMPLE PROBLEMS

1. Two smooth steel spheres moving with initial velocities shown collide with the line joining their centers in the direction of the velocity V . From previous experiments it is known that the coefficient of restitution for these conditions is 0.70. Determine the final velocity V of each sphere and the percentage loss of kinetic energy.



Solution: The equal and opposite force of contact on each sphere is along the x-direction so that the linear momentum of the system is conserved in that direction. Also, since there is no force on either sphere in the y-direction, there is no change in the y-component of either velocity. Thus:

$[\Delta\ G_x = 0]$ system : $(2 \times 10 \times 0.866) - (4 \times 5) = 4v_2' \ 2v_1' \cos\theta$,

$[\Delta\ V_y = 0]$ each sphere: $v_1' \sin\theta = 10 \times 0.5,\ v_{2_y}' = 0.$

The coefficient of restitution is the ratio of relative separation velocity to relative approach velocity both measured in the direction of the impact force. Therefore:

$$[e = \frac{\Delta v_x}{\Delta v_x}] \quad : \quad 0.70 = \frac{v_2 + v_1 \cos\theta}{5 + (10 \times 0.866)}$$

The simultaneous solution of these three equations gives:

$V' = 8.46$ ft./sec., $\theta = 36°14'$, $V = 2.74$ ft./sec. Ans.

The initial kinetic energy of the system is:

$$\frac{1}{2}.\frac{2}{32.2}(10)^2 + \frac{1}{2}.\frac{4}{32.2}.5^2 = 4.66 \text{ ft.lb.}$$

The final kinetic energy is:

$$\frac{1}{2}.\frac{2}{32.2}(8.46)^2 + \frac{1}{2}.\frac{4}{32.2}(2.74)^2 = 2.69 \text{ ft.lb.}$$

The percentage loss is:

$$\frac{4.66 - 2.69}{4.66} \times 100 = 42.3 \text{ per cent} \qquad \text{Ans.}$$

PROBLEMS

1. A steel ball is dropped from rest from a height h above a horizontal steel plate of large weight and rebounds to a height h'. Determine the coefficient of restitution e.

Ans. $e = \sqrt{\frac{h'}{h}}$

2. Cars A and B of equal weight collide at right angles at the intersection of two icy roads. The cars become entangled and move off together in the direction indicated by V', their common velocity after impact. If car A was travelling at 30 mi./hr. at the instant of impact, determine the velocity V_B of car B just before impact.

Ans. $V_B = 52.0$ mi./hr.

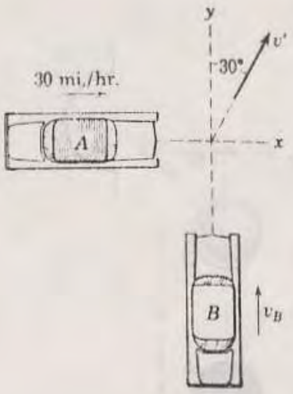


Fig. 2

3. In selecting the ram of a pile driver for a certain job it is desired that the ram will give up all of its kinetic energy at each blow. Hence velocity of the ram is zero immediately after impact. The piles to be driven weigh 800 lb. each, and experience has shown that a coefficient of restitution of 0.3 can be expected. What should be the weight W of the ram? Compute the velocity V of the pile immediately after impact if the ram is dropped from a height of 10 ft. onto the pile. Also compute the energy loss ΔE due to impact at each blow.

Ans. W = 240 lb., V = 7.61 ft./sec., ΔE = 1680 ft.lb.

4. In a low-velocity impact study a steel ball weighing 0.055 lb. is dropped from rest through a vertical distance of 6 ft. onto a 0.610 lb. steel cylinder supported by a light rod which acts as a cantilever beam. A maximum deflection of 0.250 in. from the position of static equilibrium is observed for the cylinder as a result of the impact. If a static calibration of the beam shows that an elastic deflection of 0.50 in. is produced by hanging a 12 lb. weight on the end of the beam, calculate the height h of rebound of the ball and the coefficient of restitution e which applies to these conditions.

Ans. h = 14.56 in.; e = 0.581

5. If the billiard ball B is to be sent to the pocket C by striking it with the cue ball A, determine the angle β for the rebound of the cue ball. Each ball is an identical $2\frac{1}{16}$ in. diameter ivory sphere for which the coefficient of restitution may be taken to be 0.9. Ans. $\beta = 43° 48'$

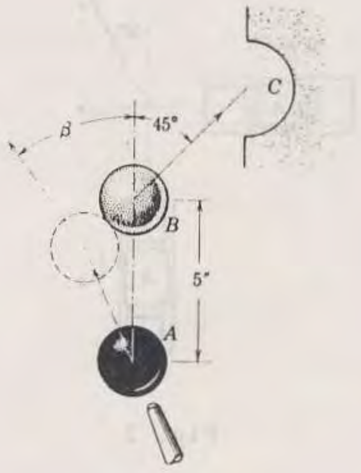


Fig. 5

20. **Gyroscopic Motion:** One of the most interesting of all problems in dynamics is that of the gyroscope which involves the rotation of a body about an axis which itself is rotating. This problem is three dimensional and may be described by the general principle of angular impulse and momentum for a rigid body with respect to a fixed point as given by Eq.(30).

Consider the rotor in Fig.16 which is pivoted at its point O with negligible frictions and which has an angular velocity of spin w about its own axis. The motion of this rotor under the action of its weight W and the pivot reaction will be described. The total angular momentum of the rotor about the fixed point O may be represented by its three vector components in the orthogonal x-y-z-directions. The z component is Hz = Iw, where I is the moment of inertia of the rotor about the z-axis and the vector stems from O. The component Hy = Iy $(\frac{d\alpha}{dt})$ is due to rotation of the rotor axis about the y-axis at the instant represented, and likewise the component Hx = Ix $(\frac{d\theta}{dt})$ is due to rotation about the x-axis. The resultant angular momentum about O is the vector sum of these three components. The resultant moment about O is Mo = Wr sin α and is in the y-direction. Eq.(30) requires that this moment be equal in magnitude and direction to the "Time rate of change of the total angular-momentum vector". Evaluation of this time rate of change where all three components of Ho are accounted for is necessary for a complete description of the motion.

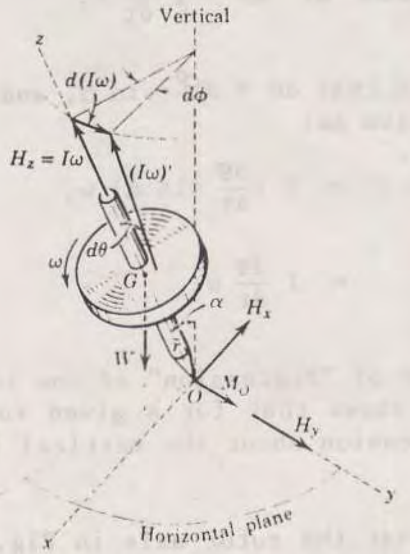


Fig. 16

The engineering aspects of the problem can be adequately explained by examining only the case where $Hz = Iw$ is very large compared with Hy and Hx. This condition occurs for a large rate of spin W. Thus only the one component of angular momentum will be considered. With this simplification Eq.(30) may be written as:

$$Mo.\ dt = d\ Ho = d\ (Iw),$$

where the moment about O is $Mo = W\bar{r}\sin\alpha$. This relation states that the change in angular momentum is equal in magnitude and direction to the applied angular impulse. This impulse has the direction of Mo, which, vectorially, is along the y-axis, and hence d (Iw) has the same direction. Thus the change in angular momentum is at right angles to the momentum or spin axis. Adding this change to Iw gives the new angular momentum (Iw)' after time dt. The momentum or spin axis of the rotor has moved through the angle dθ which is given by:

$$d\theta = \frac{d(Iw)}{Iw}$$

Combination with the preceding equation gives:

$$Mo. dt = Iwd\theta \quad or \quad Mo = I \frac{d\theta}{dt} W.$$

From the figure it is seen that $d\theta = d\varphi \sin\alpha$, and thus the resulting equation may be written also as:

$$W\bar{r} \sin\alpha = I \left(\frac{d\varphi}{dt} \sin\alpha\right) w,$$

$$\text{or} \quad W\bar{r} = I \frac{d\varphi}{dt} w,$$

where $d\varphi/dt$ is the rate of "Precession" of the rotor axis about the vertical. This relation shows that for a given rotor and given spin velocity the rate of precession about the vertical is the same for all values of α.

The reason that the rotor axis in Fig. 16 revolves about the vertical at a constant angle α instead of falling toward the x-y plane lies in the fact that the precession described is the only motion which will make the vector change in angular momentum and have the same direction as the applied moment and angular impulse. If the rotor had no spin velocity, it would indeed fall. Actually as the spin velocity decreases because of friction, the rotor axis will drop toward the horizontal plane in a rather complex manner which requires the retention of the momentum components Hx and Hy for description. In understanding the gyroscopic effect it is helpful to note that precession of the axis of spin occurs when a moment is applied whose vector is at right angles to the angular momentum vector. In the problem of plane motion, on the other hand, the moment and angular momentum vectors are parallel.

In most engineering applications of gyroscopes the moment, spin and precession axes are mutually perpendicular. This situation is illustrated in Fig. 17a where the axis of the rotor of the previous figure is now horizontal. The momentum equation becomes:

$$M = I\Omega w, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (33)$$

where Ω is the rate of precession dI/dt about the vertical and M is the moment $W\bar{r}$ of the weight about O. The moment vector M in Eq.(33) is normal to both the spin axis W and the precession axis Ω and represents the moment in this direction about the pivot O due to all forces acting on the gyro rotor. For the rotor illustrated Eq.(33) is not exact since it accounts for only the predominant momentum change in the direction of M. The error is exceedingly small, however, for the relatively high spin velocities normally used.

Equation(33) and the corresponding relationship between the senses of the three vectors M, Ω, and W will now be determined directly for the case illustrated in Fig. 17a as further aid to correct interpretation of the gyroscopic equation. Equation(30) requires that the angular impulse M.dt. during time dt about the y-axis through the fixed point O must equal the change dHz in angular momentum both in magnitude and direction. These vectors are shown in Fig. 17b, and it is seen that during the time dt the momentum or spin

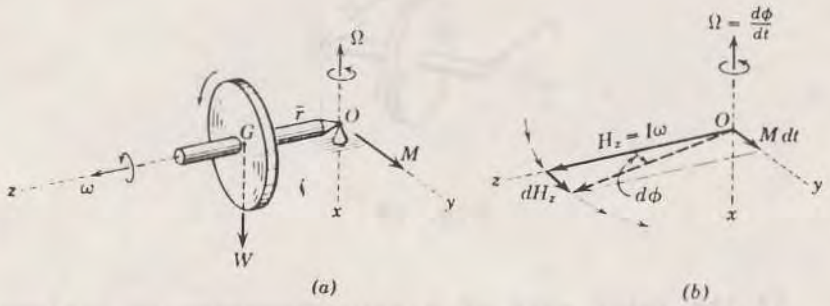


Fig. 17

axis has swung through the angle $d\varphi$ in order that dHz equal M.dt. It follows, then, that M.dt = dHz = Iw.$d\varphi$, and division by dt yields $M = I\Omega w$.

It should be carefully noted that the direction of the precession is determined by the fact that the vector change dHz in angular momentum has the same sense as the applied moment M, and hence the spin axis will always rotate toward the moment axis. The three vectors M, Ω, and w constitute a right-handed set of axes. Thus in rotating from the M-axis to the Ω-axis through the 90 deg. angle, advancement for a right-hand screw is along the w-axis. Likewise a right-hand screw would advance in the M-direction when rotated from the Ω-axis to the w-axis or would advance in the Ω-direction when rotated from the w-axis to the M-axis. Use of this right-hand rule requires memory of the sequence M-Ω-w of these vectors. This sequence can be establilshed quickly by recognizing, basically, that the momentum axis rotates toward the moment axis since the vector change in the spin momentum must have the same sense as the applied moment.

In the event an additional moment about O is applied to the rotor a corresponding additional precession will occur which obeys the rules just cited. Thus, if a force F were applied to the rotating end of the rotor shaft, Fig. 18, in the positive y-direction, the corresponding moment vector would be in the negative x-direction. The momentum or spin axis w would have a vector change in the direction of the moment axis M which is vertically up, and the spin axis would rise. Conversely, if a force were applied to the end of the rotor shaft in the direction to oppose the precession (negative y-direction) the spin axis would fall.

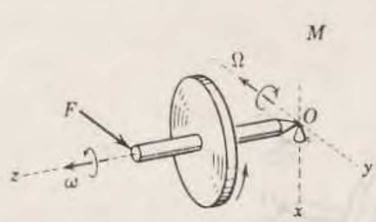


Fig. 18

If the rotor axis of a gyro precesses in a given plane, the center of gravity of the rotor remains in that plane and can have no acceleration normal to it. It follows that the resultant of all forces acting on the rotor in a plane normal to the precession plane and containing the rotor axis can not be a force and, if not zero, must be a couple. Thus the moment M in Eq.(33) is a couple and is known as the "Gyroscopic Couple". Since the value of a couple is independent of which parallel axis is chosen for evaluating its moment, a moment summation about any axis normal to the rotor axis and lying in the precession plane may be used for evaluating M. Thus for the rotor in Fig. 17a, there is no vertical acceleration of G during the horizontal precession, and the upward force exerted by the pivot on the rotor at O. equals the weight W and together with the weight W constitute the gyroscopic couple $W\bar{r}$. The correct value of the couple may be obtained by taking moments about an axis parallel to the y-axis through O, G, or any other point.

When w becomes small for the rotor of Fig. 17a, the axis begins to droop, and Eq.(33) is no longer a good approximation. On the other hand, if the rotor axis is confined to rotate in the horizontal plane about the vertical by some type of restraining guides, there can be no angular momentum about the y-axis at all, and the only change in angular momentum in the y-direction comes from the directional change in Hz = Iw as expressed by Eq.(33). Therefore Eq.(33) is exact whenever the axis of a symmetrical rotor is confined to precess exclusively in one plane. If the end of the rotor axis in Fig. 18 were constrained by smooth guides (not shown) to move only in the horizontal y-z plane, then the force F would cause accelerated rotation of the shaft about the x-axis given by $F\ell = I_x\alpha$ where ℓ is the moment arm to O, α is the angular acceleration of the shaft axis about the vertical, and Ix is the moment of inertia of the rotor about the vertical. Accompanying this rotation there would be a downward force exerted by the guide on the shaft whose moment M about O obeys Eq.(33) exactly at any instant.

In addition to being a toy the gyroscope has important engineering application. First, it is used extensively as a directional device. With a mounting in gimbal rings, Fig. 19, the gyroscope is free from external moments, and its axis will retain a fixed direction irrespective of the rotational movement of its base. Independence from the rotational movement of the surrounding is used

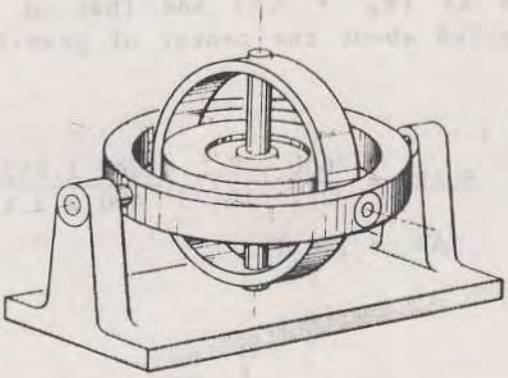


Fig. 19

as a positioning control device. By adding a pendulous weight to the inner gimbal ring the attraction of the earth may be used to cause precession of the gyro so that the spin axis always points north. This action forms the basis of the gyro compass. Gyroscopic action also forms the basis for inertial-guidance systems. The gyroscope has found important use as a stabilizing device. The controlled precession of a large gyro mounted in a ship is used to produce a moment to counteract the rolling of the ship at sea. The gyroscopic effect is an extremely important consideration in the design of bearings for the shafts of rotors subject to forced precession.

SAMPLE PROBLEM

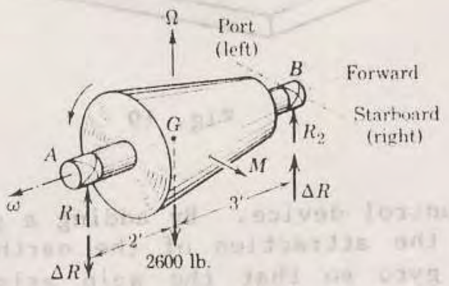
1. The turbine rotor in a ship's power plant weighs 2600 lb. with center of gravity at G and has a radius of gyration of 8 in. The rotor is mounted in bearings A and B with its axis in the horizontal fore-and-aft direction and turns counterclockwise at 5000 rev./min. when viewed from the stern. Determine the vertical components of the bearing reactions at A and B if the ship is making a turn to port (left) of 400 yard radius at a speed of 22 knots (1 knot = 1.152 mi./hr.).

Solution: The vertical components of the bearing reactions will equal the static reactions R_1 and R_2 plus or minus the increment ΔR due to the gyroscopic effect. The moment principle easily gives R_1 = 1560 lb. and R_2 = 1040 lb. The direction of the spin velocity w and the precessional velocity Ω are indicated with the free-body diagram. Since the spin axis tends to rotate toward the torque axis, the gyroscopic couple M due to the ΔR's points to starboard, and thus the reaction at B is $(R_2 + \Delta R)$ and that at A is $(R_1 - \Delta R)$. Equation(33) is applied about the center of gravity of the rotor and gives:

$[M = I\Omega w]$:

$$5.\Delta R. = \frac{2600}{32.2}\left(\frac{8}{12}\right)^2 \left(\frac{22 \times 1.152 \times 44}{400 \times 3 \times 30}\right)\left(\frac{5000 \times 2\pi}{60}\right)$$

$$\Delta R = 116 \text{ lb.}$$



The required reactions are then:

$A = 1560 - 116 = 1444$ lb.; $B = 1040 + 116 = 1156$ Ans.

The horizontal components of the bearing reactions necessary to give the rotor its centripetal acceleration in the turn may be computed, and each total bearing reaction determined if desired.

PROBLEMS

1. One type of aircraft-engine supercharger consists of the 4.10 lb. blower A with a radius of gyration of 2.90 in. which is driven at 18,000 rev./min. by the 12.20 lb. exhaust turbine B with a radius of gyration of 2.75 in. Determine the radial forces on the bearing C and D if the shaft is mounted in a vertical position and the airplane is rolling (turning about the horizontal flight axis) at the rate of 3 rad./sec. Ans. /C/ = /D/ = 232 lb.

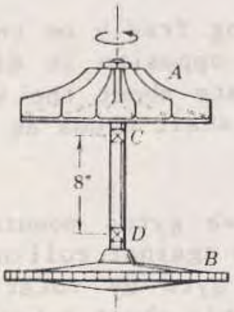


Fig. 1

Fig. 2

2. An airplane has a take-off speed of 150 mi./hr. and retracts its main landing gear in the manner shown. Each of the two 36 in. diameter wheels weighs 74 lb. and has a radius of gyration of 12 in. If the retracting gear folds into the wing with an angular velocity of 0.5 rad./sec., find the added bending moment M in the wheel bearing due to the gyroscopic action with the wheels still spinning at the take-off speed.

3. The 400 lb. rotor for a turbojet engine has a radius of gyration of 10 in. and rotates clockwise at 15,000 rev./min. when viewed from the front of the airplane. If the air plane is travelling at 600 mi./hr. and making a turn to the right of 2 mi. radius, compute the gyroscopic moment M which the rotor bearings must support. Does the nose of the airplane tend to rise or fall as a result of the gyroscopic action?

Ans. M = 1130 lb.ft., nose tends to rise.

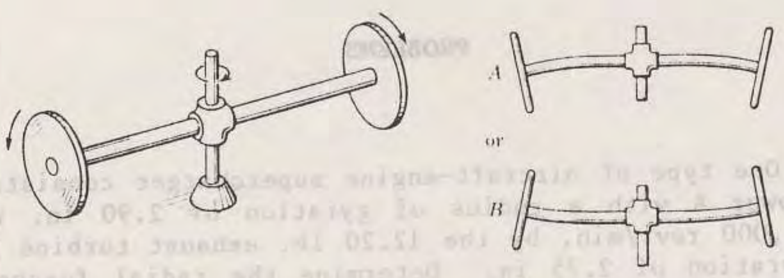


Fig. 4

4. The two identical disks are rotating freely on the shaft with angular velocities equal in magnitude and opposite in direction as shown. The shaft in turn is caused to rotate about the vertical axis in the sense indicated. Prove whether the shaft bends as in A or as in B. Ans. A.

5. In the figure is shown one of three gyros mounted with vertical axis and used to stabilize a large ship against rolling. The motor A turns the pinion which precesses the gyro by rotating the large precession gear B and attached rotor assembly about a horizontal

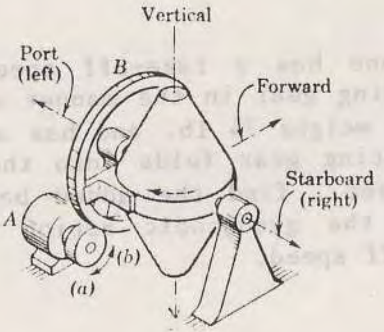


Fig. 5

transverse axis in the ship. The rotor turns inside the housing at a clockwise speed of 800 rev./min. when viewed from the top and has a weight of 100 tons with a radius of gyration of 4.85 ft. Determine the moment exerted on the hull structure by the gyro if the motor turns the precession gear at the rate of 0.420 rad./sec. In which of the two directions, (a) or (b), should the motor turn to counteract a roll of the ship to starboard?

Ans. $M = 5.14 \times 10^6$ lb.ft.; (a).

MOMENTS OF INERTIA

I. MOMENTS OF INERTIA OF AREAS:

There are many problems in engineering which involve the evaluation of an integral of the form $\int y^2 DA$ where y is the distance from an element of area dA to an axis which is either in or normal to the plane of the area. Integrals of this form appear so frequently that it is of great advantage to develop them for some of the more common areas and to tabulate the results for easy reference.

Figure Cl. illustrates the physical origin of these integrals. In the a part of the figure the surface area ABCD is subjected to a distributed pressure p whose intensity is proportional to the distance y from the axis AB. This situation covers the discussion on "Pressure on Submerged Surfaces", and describes the action of liquid pressure on a plane surface. The moment about AB that is due to the pressure on the element of area dA is pydA. = ky^2fA. Thus the integral in question appears when the total moment $M = k\int y^2 dA$. is evaluated. In part (b) of Fig. Cl. is shown the distribution of stress acting on a transverse section of a simple elastic beam bent by equal and opposite couples applied to its ends. At any section of the beam a linear distribution of force intensity or stress, given by $\sigma = ky$, is present. The elemental moment about the axis 0-0 is $dM = \sigma.y.dA = ky^2 dA$. Thus the same integral appears when the total moment $M = k\int y^2 .dA$. is evaluated. A third example is given in part (c) of Fig. Cl. which shows a circular shaft subjected to a twist or torsional moment. Within the elastic limits of the material this moment is resisted at each cross-section of the shaft by a distribution of tangential or shear stress τ which is propostional to the radial distance r from the center. Thus $\tau = kr$, and the total moment about the central axis is $M = \int \tau.r.dA = k\int r^2.dA$. Here the integral differs from the preceding two examples in that the area is normal instead of parallel to the moment axis and that r is a radial coordinate instead of a rectangular one.

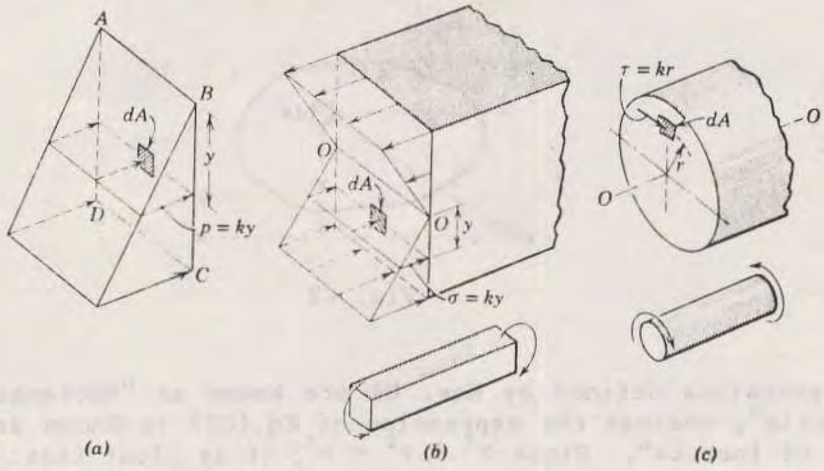


Fig. C1

The integral just illustrated is generally called the "Moment of Inertia" of the area about the axis in question. A more fitting term is the "Second Moment of Area" since the first moment y.dA. is multiplied again by the moment arm y to obtain the result for the element dA. The word "Inertia" appears in the terminology by reason of the similarity between the mathematical form of the integrals for second moments of areas and those for the resultant moments of the so-called inertia forces in the case of rotating bodies. The moment of inertia of an area is a purely mathematical property of the area and in itself has no physical significance.

Consider the area A in the x-y plane, Fig. C2. The moments of inertia of the element dA about the x- and y- axes are, by definition, $dIx = y^2.dA.$ and $dIy = x^2.dA.$, respectively. Therefore the moments of inertia of A about the same axis are:

$$Ix = \int y^2.dA, \qquad Iy = \int x^2.dA, \qquad \text{(C1)}$$

where the integration covers the entire area. The moment of inertia of dA. about the pole O (z-axis) is, by similar definition, dJz = .dA, and the moment of inertia of the entire area about O is:

$$Jz = \int r^2.dA \qquad \text{(C2)}$$

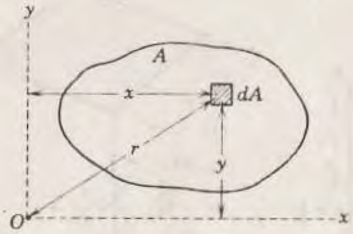


Fig. C2

The expressions defined by Eqs. C1 are known as "Rectangular Moments of Inertia", whereas the expression of Eq.(C2) is known as the "Polar Moment of Inertia". Since $x^2 + y^2 = r^2$, it is clear that:

$$Jz = Ix + Iy \quad \text{.........................(C3)}$$

A polar moment of inertia of an area whose boundaries are more simply described in rectangular coordinates than in polar coordinates is easily calculated with the aid of Eq. C3.

It should be noted that the moment of inertia of an element involves the square of the distance from the inertia axis to the element. An element whose coordinate is negative contributes as much to the moment of inertia as does an element with a positive coordinate of the same magnitude. Consequently, the moment of inertia of an area about any axis is always a positive quantity. In contrast, the first moment of the area, which was involved in the computations of centroids, could be either positive or negative.

The dimensions of inertia of area are clearly L^4, where L stands for the dimension of length. Thus the units for area moments of inertia are expressed as quartic inches ($in.^4$) or quartic feet ($ft.^4$).

The choice of the coordinates to use for the calculation of moments of inertia is important. Rectangular coordinates should be used for shapes whose boundaries are most easily expressed in these coordinates. Polar coordinates will usually simplify problems involving boundaries which are easily described in r and θ. The choice of an element of area which simplifies the integration as much as posssible is also important.

Radius of Gyration: The moment of inertia of an area is a measure of the distribution of the area from the axis in question.

Assume all the area A, Fig. C3, to be concentrated into a strip of negligible thickness at a distance kx from the x-axis so that the product $k_x^2 A$ equals the moment of inertia about the axis. The distance kx, called the "Radius of Gyration", is then a measure of the distribution of area from the inertia axis. By definition, then, for any axis:

$$I = k^2 A \text{ or } k = \sqrt{\frac{I}{A}} \quad \text{................. (C4)}$$

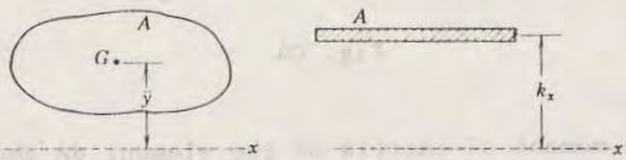


Fig. C3

When this definition is substituted in each of the three terms in Eq. C3, there results:

$$k_z^2 = k_x^2 + k_y^2 \quad \text{........................ (C5)}$$

Thus the square of the radius of gyration about a polar axis equals the sum of the squares of the radii of gyration about the two corresponding rectangular axes.

It is imperative that there be no confusion between the coordinate $\bar{y}$ to the centroid of the area and the radius of gyration k. The square of the centroid distance, Fig. C3, is $\bar{y}^2$ and is the square of the mean value of the distances y from the elements dA to the axis. The quantity k_x^2, on the other hand, is the mean of the squares of these distances. The moment of inertia is not equal to since the square of the mean is not equal to the mean of the squares.

Transfer of Axes: The moment of inertia of an area about a noncentroidal axis may be easily expressed in terms of the moment of inertia about a parallel centroidal axis. In Fig. C4, the Xo - Yo axes pass through the centroid G of the area. Let it be desired to determine the moments of inertia of the area about the parallel x-y axes.

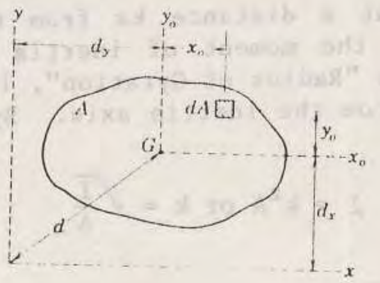


Fig. C4

By definition the moment of inertia of the element dA about the x-axis is:

$$dIx = (Yo + dx)^2 .dA.$$

Expanding and integrating give:

$$Ix = \int y_o^2 .dA + 2d_x \int y_o dA + d_x{}^2 \int dA.$$

The first integral is the moment of inertia $\bar{I}x$ about the centroidal Xo-axis. The second integral is zero since $A\bar{y}o = \int yo.dA$ and Yo is automatically zero. The third integral is simply Ad^2x. Thus the expression for Ix and the similar expression for Iy become:

$$Ix = \bar{I}x + A.d^2x.$$
$$Iy = \bar{I}y + A.d^2y. \quad \text{.................... (C6)}$$

By Eq. C3 the sum of these two equations gives:

$$Jz = \bar{J}z + Ad^2 . \quad \text{...................... (C6a)}$$

Equations C6 and C6a are the so-called "Parallel-Axis Theorems". Two points in particular should be noted. First, the axes between which the transfer is made must be parallel, and, second, one of the axes must pass through the centroid of the area.

If a transfer is desired between two parallel axes neither one of which passes through the centroid, it is first necessary to transfer from one axis to the parallel centroidal axis and then to transfer from the centroidal axis to the second axis.

The parallel-axis theorems also hold for radii of gyration. With substitution of the definition of k into Eqs. C6, the transfer relation becomes:

$$k^2 = \bar{k}^2 + d^2 \quad \text{.....................} \quad (C6b)$$

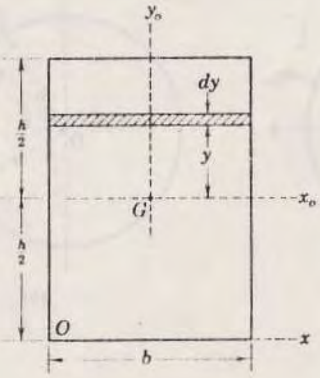
where $\bar{k}$ is the radius of gyration about a centroidal axis parallel to the axis about which k applies and d is the distance between the two axes. The axes may be either in the plane or normal to the plane of the area.

SAMPLE PROBLEM

1. Determine the moments of inertia of the rectangular area about the centoidal Xo - Yo axes, the centroidal polar axis G, the x-axis, and the polar axis O.

Solution: For the calculation of the moment of inertia $\bar{I}x$ about the Xo-axis a horizontal strip of area b.dy. is chosen so that all elements of the strip have the same y-coordinate. Thus:

$$[Ix = \int y^2.dA] : \quad \bar{I}x = \int_{-\frac{h}{2}}^{+\frac{h}{2}} y^2.b.dy = \frac{1}{12} bh^3 \qquad \underline{\text{Ans.}}$$



By interchanging symbols the moment of inertia about the centroidal Yo-axis is:

$$Iy = \frac{1}{12} hb^3 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \underline{Ans.}$$

The centroidal polar moment of inertia is:

$$[Jz = Ix + Iy]: \quad \bar{J}z = \frac{1}{12}(bh^3 + b^3h) = \frac{1}{12} A (b^2 + h^2) \quad \underline{Ans.}$$

By the parallel axis theorem the moment of inertia about the x-axis is:

$$[Ix = \bar{I}x + Adx^2]: \quad Ix = \frac{1}{12} bh^3 + bh\left(\frac{h}{2}\right)^2 = \frac{1}{3} bh^3 = \frac{1}{3} Ah^2 \quad \underline{Ans.}$$

The polar moment of inertia about O may also be obtained by the parallel axis theorem. Thus:

$$[Jz = \bar{J}z + Ad\ ]: \quad Jz = \frac{1}{12} A(b^2 + h^2) + A\left[\left(\frac{b}{2}\right)^2 + \left(\frac{h}{2}\right)^2\right]$$

$$Jz = \frac{1}{3} A(b^2 + h^2) \quad \ldots\ldots\ldots \underline{Ans.}$$

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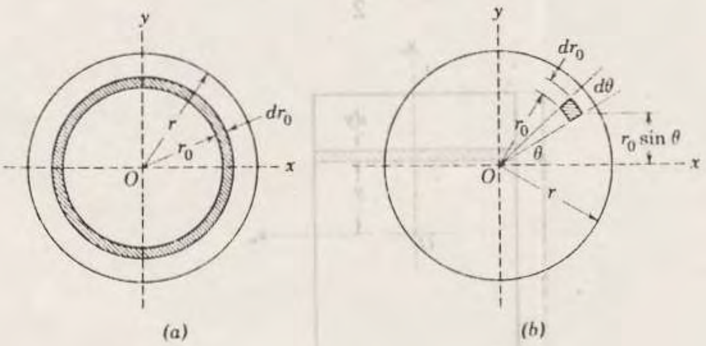
2. Calculate the moments of inertia of the area of the circle about a diametral axis and about the polar axis through the center. Specify the radii of gyration.



Solution: An element of area in the form of a circular ring, shown in part (a) of the figure, may be used for the calculation of the moment of inertia about the polar z-axis through O since all elements of the ring are equidistant from O. The elemental area is $dA = 2\pi r_o dr_o$, and thus:

$$[Jz = \int r^2 . dA] : \quad Jz = \int_o^r r^2{}_o \, (2\pi r_o . dr_o) = \frac{\pi r^4}{2} = \frac{1}{2} Ar^2 \qquad \underline{\text{Ans.}}$$

The polar radius of gyration is:

$$[k = \sqrt{\frac{I}{A}}] : \quad kz = \frac{r}{\sqrt{2}} \qquad \underline{\text{Ans.}}$$

By symmetry $Ix = Iy$, so that from Eq. C3,

$$[Jz = Ix + Iy]: \; Ix = \frac{1}{2} Jz = \frac{\pi r^4}{4} = \frac{1}{4} Ar^2 \qquad \underline{\text{Ans.}}$$

The radius of gyration about the diametral axis is:

$$[k = \sqrt{\frac{I}{A}}] : \quad kx = \frac{r}{2} \qquad \underline{\text{Ans.}}$$

The foregoing determination of Ix is the simplest possible. The result may also be obtained by direct integration, using the element of area $dA = ro.dro.d\theta$ shown in part (b) of the figure. By definition:

$$[Ix = \int y^2 dA] : \quad Ix = \int_o^{2\pi}\int_o^{r} (ro \sin\theta)^2 \, ro \, dro \, d\theta,$$

$$= \int_o^{2\pi} \frac{r^4 \sin^2\theta}{4} d\theta$$

$$= \frac{r^4}{4} . \frac{1}{2} [\theta - \frac{\sin 2\theta}{2}]_o^{2\pi} = \frac{\pi r^4}{4}$$

---

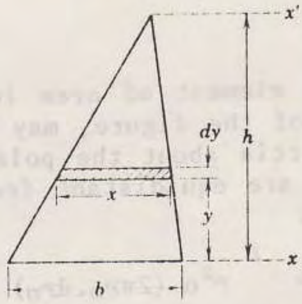
3. Determine the moments of inertia of the triangular area about its base and about prallel axes through its centroid and vertex.

Solution: A strip of area parallel to the base is selected as shown in the figure and it has the area $dA = x.dy = [(h-y)\frac{b}{h}] \, dy$. By definition:

$$[Ix = \int y^2 dA]:$$

$$Ix = \int_o^h y^2 \frac{h-y}{h} b.dy = b \, [\frac{y^3}{3} - \frac{y^4}{4h}]_o^h$$

$$= \frac{bh^3}{12}$$



By the parallel-axis theorem the moment of inertia I about an axis through the centroid, a distance $\frac{h}{3}$ above the x-axis, is:

$$[\bar{I} = I - Ad^2] : \quad \bar{I} = \frac{bh^3}{12} - (\frac{bh}{2})(\frac{h}{3})^2 = \frac{bh^3}{36} \qquad \underline{\text{Ans}}.$$

A transfer from the centroidal axis to the x'-axis through the vertex gives:

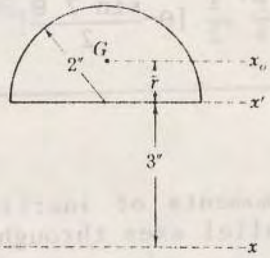
$$[I = \bar{I} + Ad^2] : \quad Ix' = \frac{bh^3}{36} + (\frac{bh}{2})(\frac{2h}{3})^2 = \frac{bh^3}{4} \qquad \underline{\text{Ans}}.$$

---

4. Determine the moment of inertia about the x-axis of the semicircular area.

Solution: The moment of inertia of the semicircular area about the x'-axis is one half of that for a complete circle about the same axis. Thus from the results of problem 2.

$$Ix' = \frac{1}{2}\frac{\pi r^4}{4} = \frac{2^4\pi}{8} = 2\pi \text{ in.}^4$$



The moment of inertia $\bar{I}$ about the parallel centroidal axis Xo is obtained next. Transfer is made through the distance $\bar{r} = \frac{4r}{3\pi} = (4 \times 2)/3\pi = 8/3\pi$ in. by the parallel-axis theorem. Hence: $[\bar{I} = I - Ad^2]$: $\bar{I} = 2\pi - (\frac{2^2\pi}{2})(\frac{8}{3\pi})^2 = 1.755$ in.^{4}. Finally, transfer is made from the centroidal Xo-axis to the x-axis, which gives:

$$[I = \bar{I} + Ad^2] : \qquad Ix = 1.755 + (\frac{2^2\pi}{2})\left(3 + \frac{8}{3\pi}\right)^2,$$

$$= 1.755 + 23.1 = 94.9 \text{ in.}^4 \qquad \underline{\text{Ans}}.$$

PROBLEMS

1. Find the moments of inertia of the area of the quarter circle shown about the diametral x-axis and the polar axis through O.

<u>Ans</u>. $Ix = \frac{\pi r^4}{16}$, $J_z = = \frac{\pi r^4}{8}$

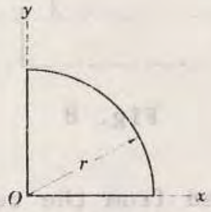


Fig. 1

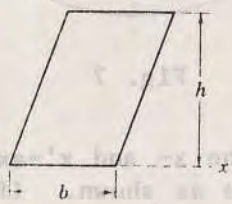


Fig. 2

2. Find the moments of inertia of the area of the parallelogram about the base (x-axis) and about a parallel centroidal axis.

3. Find the radius of gyration k of a square of side b about one diagonal.

<u>Ans</u>. $k = \frac{6}{2\sqrt{3}}$

4. Find the moment of inertia of the shaded area about the x-axis by taking a horizontal strip of area dA.

<u>Ans</u>. $Ix = \frac{4}{15} ab^3$.

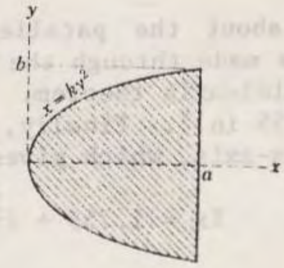


Fig. 4

5. Solve problem 4. by choosing a vertical strip of area dA.

6. Find the moment of inertia of the figure in problem 4. about the y-axis. <u>Ans</u>. $I_y = \frac{4}{7} a^3 b$.

7. Calculate the moments of inertia of the elliptical area about the major axes and the central polar axis.

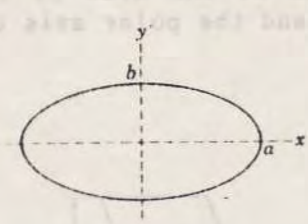


Fig. 7

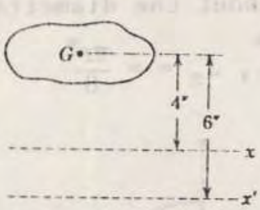


Fig. 8

8. The x- and x'-axes are located from the centroid G of the irregular area as shown. If the moments of inertia about these axes are $I_x = 1800$ in.^{4} and $I_{x'} = 2200$ in.^{4}, determine the area A of the figure. <u>Ans</u>. $A = 20$ in.^{2}

9. Determine the moments of inertia of the area of the circular sector about the x- and y-axes.

$$\underline{\text{Ans}}.\quad I_x = \frac{r^4}{4}\left(\alpha - \frac{\sin 2\alpha}{2}\right),\; I_y = \frac{r^4}{4}\left(\alpha + \frac{\sin 2\alpha}{2}\right)$$

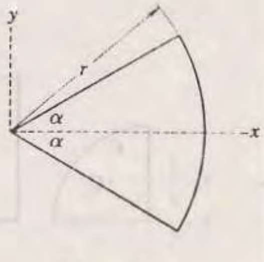


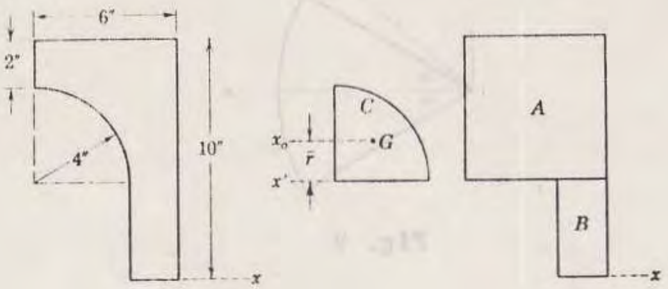
Fig. 9

Composite Areas: The moment of inertia of composite area about a particular axis is the algebraic sum of the moments of inertia of the various parts about the same axis. The results of the problems proved previously may be used to determine the moments of inertia for component parts of the shapes given. It is often convenient to regard a composite area as composed of positive and negative parts. The moment of inertia of a negative area is a minus quantity.

When section is composed of a large number of parts, it is convenient to tabulate the results for the parts in terms of the area A, centroidal moment of inertia $\bar{I}$, distance d from the centroidal axis to the axis about which the moment of inertia of the entire section is being computed, and the product Ad^2. For any one of the parts the desired moment of inertia is $\bar{I} + Ad^2$, and thus for the entire section the desired moment of inertia may be expressed as $I = \Sigma\bar{I} + \Sigma Ad^2$.

SAMPLE PROBLEM

1. Compute the moment of inertia and radius of gyration about the x-axis for the cross section shown.



Solution: The composite area may be considered as composed of the two rectangles A and B and the negative quarter circular area C. For the rectangle A the moment of inertia about the x-axis is:

$$[I = \bar{I} + Ad^2] : \quad Ix = (\frac{1}{12} \times 6 \times 6^3) + (6^2 \times 7^2) = 1872 \text{ in.}^4$$

The moment of inertia of B about the x-axis is:

$$Ix = \frac{1}{3} \times 2 \times 4^3 = 42.67 \text{ in.}^4$$

The moment of inertia of the negative quarter circle C about its horizontal diameter is:

$$Ix = -\frac{1}{4} \times \frac{1}{4}\pi \times 4^4 = -50.27 \text{ in.}^4$$

Transfer of this result through the distance $\bar{r} = \frac{4\pi}{3\pi} = (4 \times 4)/3\pi =$ 1.697 in. gives for the centroidal moment of inertia of C:

$$[\bar{I} = I - Ad^2] : \quad I = -50.27 - (-\frac{\pi}{4} \times 4^2)\ (1.679)^2 = -14.07 \text{ in.}^4$$

The moment of inertia of C may now be found with respect to the x-axis, and the transfer from the centroidal axis gives:

$$[I = \bar{I} + Ad^2]: \quad Ix = -14.07 + (-\frac{\pi}{4} \times 4^2)\ (4 + 1.697)^2 = -422 \text{ in.}^4$$

The moment of inertia of the net section about the x-axis is the sum of moments of inertia of its component parts. Thus:

$$Ix = 1872 + 42.7 - 422 = 1493 \text{ in.}^4 \quad \ldots\ldots\ldots\ldots \underline{\text{Ans.}}$$

$$kx = \sqrt{\frac{Ix}{A}} = \sqrt{\frac{1493}{31.43}} = = 6.89 \text{ in.} \quad \ldots\ldots\ldots\ldots \underline{\text{Ans.}}$$

PROBLEMS

1. Determine the polar moment of inertia J for the section about point O. Ans. J = 138.9 in.

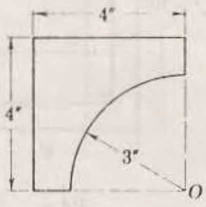


Fig. 1

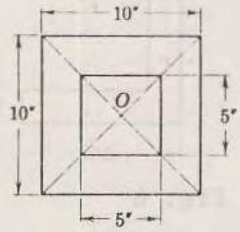


Fig. 2

2. Find the polar moment of inertia J about point O for the cross-section bound by the two squares.

3. Find the polar moment of inertia of the net area about O.

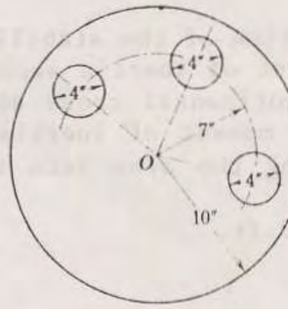


Fig. 3

4. Determine the moment of inertia of the area of a rectangle of side a and b about a diagonal.

$$\underline{\text{Ans}}.\quad I = \frac{a^3\ b^3}{6(a^2 + b^2)}$$

5. Find the moment of inertia about the x-axis of the area between the curves x = y and x = y from x = 0 to x = 1, where x and y are in inches.

6. Determine the moments of inertia of the Z-section about the centroidal Xo-and Yo-axes.

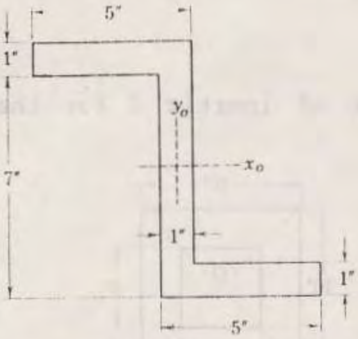


Fig. 6

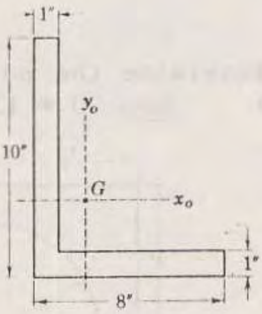


Fig. 7

7. Determine the moment of inertia of the angle section about its horizontal centroidal axis Xo. Ans. $\overline{I}x = 167.3$ in.

8. Determine the polar moment of inertia J of the hexagonal area about its centroid. The length of one side is b.

Ans. $J = \frac{5\sqrt{3}}{8} b^4$.

9. In the calculation of the stability of a ship's hull it is necessary to know the moment of inertia about the longitudinal center line of the area of the horizontal cross section of the hull at the waterline. Estimate this moment of inertia for the waterline shape reproduced here by dividing the area into a number of approximating strips.

Ans. $I \doteq 53{,}000$ ft.^{4}

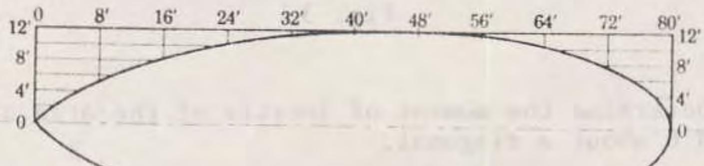


Fig. 9

10. Determine the moment of inertia about the x-axis of the cross section shown. Ans. $Ix = 1611$ in.^{4}

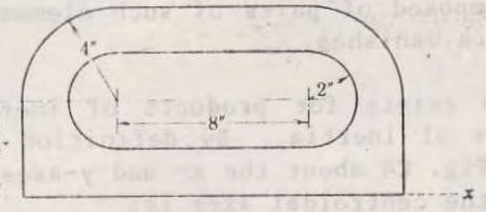


Fig. 10

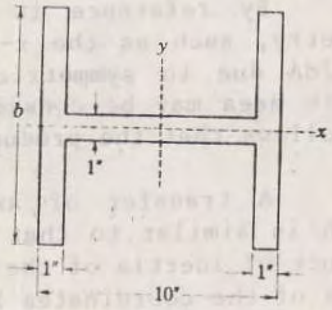


Fig. 11

11. Determine the flange width b for the H-beam section so that the moments of inertia about the central x- and y-axes will be equal. Ans. b = 16.1 in.

Product of Inertia: In certain problems involving unsymmetrical cross sections an expression occurs which has the form:

$$d\,Pxy = xy\,dA, \qquad Pxy = \int xy\,dA, \quad \text{.....................} \quad (C7)$$

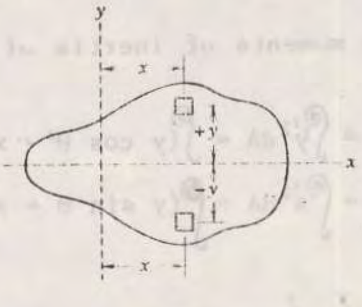


Fig. C5

where x and y are the coordinates of the element of area dA. The quantity Pxy is called the "Product of Inertia" of the area A about the x-y axes. Unlike moments of inertia, the product of inertia can be positive or negative.

By reference to Fig. C5 it can be seen for an axis of symmetry, such as the x-axis, that the sum of the terms $x(-y)dA$ and $x(+y)dA$ due to symmetrically placed elements vanishes. Since the entire area may be considered as composed of pairs of such elements, it follows that the product of inertia vanishes.

A transfer of axis theorem exists for products of inertia which is similar to that for moments of inertia. By definition the product of inertia of the area A in Fig. C4 about the x- and y-axes in terms of the coordinates Xo, Yo, to the centroidal axes is:

$$P_{xy} = \int_A (X_o + d_y)(Y_o + d_x)\, dA,$$

$$= \int_A X_oY_o dA + d_x \int_A X_o dA + d_y \int_A Y_o dA + d_x d_y \int_A dA,$$

$$P_{xy} = \bar{P}_{xy} + d_x d_y A, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (C8)$$

where $\bar{P}_{xy}$ is the product of inertia with respect to the centroidal Xo - Yo axes which are parallel to the x-y axes.

Inclined Axes: It is often necessary to calculate the moment of inertia of an area about inclined axes. This consideration leads directly to the important problem of determining the axes about which the moment of inertia is a maximum and a minimum.

In Fig. C6 the moments of inertia of the area about the x'- and y'- axes are:

$$I_{x'} = \int_A y'^2 dA = \int_A (y \cos\theta - x \sin\theta)^2\, dA,$$

$$I_{y'} = \int_A x'^2 dA = \int_A (y \sin\theta + x \cos\theta)^2\, dA.$$

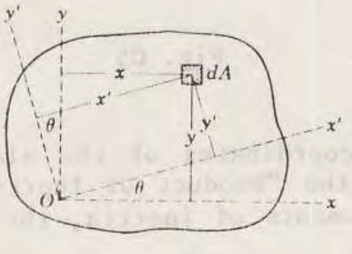


Fig. C6

Expanding and substituting the trigonometric identities, $\sin^2\theta = \frac{1-\cos 2\theta}{2}$, $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, and the defining relations for Ix, Iy, Pxy, give:

$$Ix' = \frac{Ix + Iy}{2} + \frac{Ix - Iy}{2}\cos 2\theta - Pxy \sin 2\theta, \quad \ldots\ldots\ldots\ldots\ (C9)$$

$$Iy' = \frac{Ix + Iy}{2} - \frac{Ix - Iy}{2}\cos 2\theta + Pxy \sin z\theta,$$

In a similar manner:

$$Pxy = \int x'.y'.dA = \frac{Ix - Iy}{2}\sin 2\theta + Pxy \cos 2\theta \ \ldots\ldots\ldots\ (C9a)$$

Adding Eqs. C9 gives $Ix' + Iy' = Ix + Iy = Jz$, the polar moment of inertia about O, which checks the result of Eq. C3.

The angle which makes Ix' and Iy' a maximum or a minimum may be determined by setting the derivative of either Ix' or Iy' with respect to O equal to zero. Thus:

$$\frac{dIx'}{dO} = (Iy - Ix)\sin 2\theta - 2\ Pxy \cos 2\theta = 0$$

Denoting this critical angle by α gives:

$$\tan 2\alpha = \frac{2\ Pxy}{Iy - Ix} \quad \ldots\ldots\ldots\ldots\ldots\ (C10)$$

Equation C10 gives two values for 2α which differ by π since $\tan 2\alpha = \tan(2\alpha + \pi)$. Consequently the two solutions for α will differ by $\frac{\pi}{2}$. One value defines the axis of maximum moment of inertia, and the other value defines the axis of minimum moment of inertia.

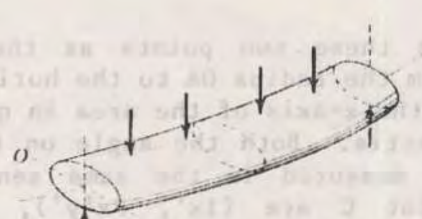


Fig. C7

These two rectangular axes are known as the "Principal Axes of Inertia". Substitution of Eq. C10 in Eq. C9a shows that the product of inertia is zero for principal axes of inertia. A beam of oval cross-section loaded transversely, Fig. C7, if free to rotate about its longitudinal axis, will turn until the horizontal axis of its cross section is the minimum axis of inertia O-O.

The relations in Eqs. C9, C9a, and C10 may be represented graphically by a diagram known as Mohr's circle. For given values of Ix, Iy, and Pxy the corresponding values of Ix', Iy', and Px'y' may be determined from the diagram for any desired angle θ. A horizontal axis for the measurements of moments of inertia and a vertical axis for the measurement of products of inertia are first selected, Fig. C8. Next, point A, which has the coordinates (Ix, Pxy), and point B, which has the coordinates (Iy, -Pxy), are located.

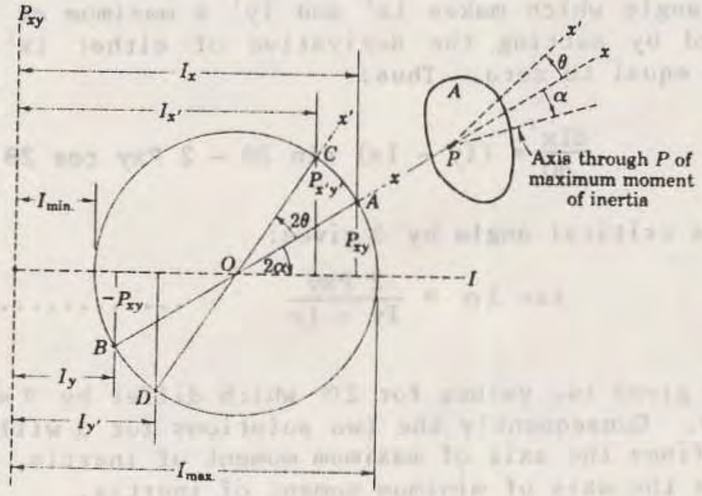


Fig. C8

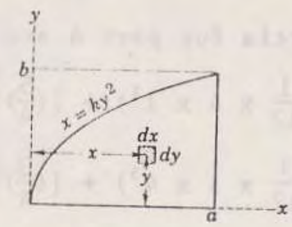
A circle is drawn with these two points as the extremities of a diameter. The angle from the radius OA to the horizontal axis is 2α or twice the angle from the x-axis of the area in question to the axis of maximum moment of inertia. Both the angle on the diagram and the angle on the area are measured in the same sense as shown. The coordinates of any point C are (Ix', Px'y'), and those of the corresponding point D are (Iy', - Px'y'). Also the angle between OA and OC is 2θ or twice the angle from the x-axis to the x'-axis. Again both angles are measured in the same sense as shown. It may be verified from the trigonometry of the circle that Eqs. C9, C9a, and C10 agree with the statements made.

SAMPLE PROBLEMS

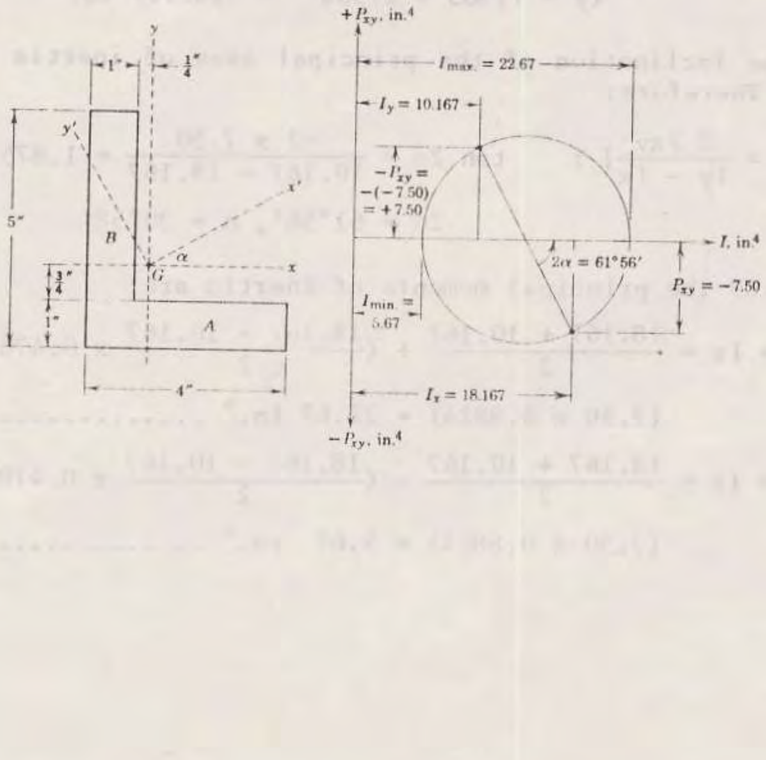
1. Determine the product of inertia for the area under the parabola shown:

Solution: The equation of the curve becomes $x = ay^2/b^2$. The product of inertia for the element $dA = dx.dy$. is $dPxy = xydx.dy$ and for the entire area is:

$$Pxy = \int_o^b \int_{ay^2/b^2}^a x.y.dx.dy. = \int_o^b \frac{1}{2}\left(a^2 - \frac{a^2y^4}{b^4}\right) y.dy = \frac{1}{6}a^2b^2 \qquad \underline{\text{Ans.}}$$



2. Locate the principal centroidal axes of inertia with their corresponding maximum and minimum moments of inertia for the angle section.



Solution: The centroid G is easily located as shown. The product of inertia for each rectangle about its own centroidal axes parallel to the x- and y-axes is zero by symmetry. Thus the product of inertia for part A is:

$[Pxy = \bar{P}xy + d_x d_y A]$: $Pxy = 0 + (-\frac{5}{4})\ (+\frac{3}{4})\ (4) = -3.75$ in.^{4}

Likewise for B,

$[Pxy = \bar{P}xy + d_x d_y A]$: $Pxy = 0 + (\frac{5}{4})\ (-\frac{3}{4})\ (4) = -3.75$ in.^{4}

For the complete angle: $Pxy = -3.75 - 3.75 = -7.50$ in.^{4}

The moments of inertia for part A are:

$[I = \bar{I} + Ad^2]$: $Ix = (\frac{1}{12} \times 4 \times 1^3) + [(\frac{5}{4})^2 \times 4] = 6.583$ in.^{4}

$Iy = (\frac{1}{12} \times 1 \times 4^3) + [(\frac{3}{4})^2 \times 4] = 7.583$ in.^{4}

In similar manner the moments of inertia for part B are Ix = 11.583 in.^{4}, Iy = 2.583 in^4. Thus for the entire section:

$Ix = 6.583 + 11.583 = 18.167$ in.^{4}

$Iy = 7.583 + 2.583 = 10.167$ in.^{4}

The inclination of the principal axes of inertia is given by Eq. C10. Therefore:

$[\tan 2\alpha = \frac{2\ Pxy}{Iy - Ix}]$: $\tan 2\alpha = \frac{-2 \times 7.50}{10.167 - 18.167} = 1.875$

$2\alpha = 61°56',\ \alpha = 30°58'$ Ans.

From Eqs. C9 the principal moments of inertia are:

$I\ max. = Ix = \frac{18.167 + 10.167}{2} + (\frac{18.167 - 10.167}{2} \times 0.4705) +$

$(7.50 \times 0.8824) = 22.67$ in.^{4} .... Ans.

$I\ min. = Iy = \frac{18.167 + 10.167}{2} - (\frac{18.167 - 10.167}{2} \times 0.4705) -$

$(7.50 \times 0.8824) = 5.67$ in.^{4} .... Ans.

These results may also be obtained graphically by construction of the Mohr circle as shown to the right of the angle in the figure.

PROBLEMS

1. Determine the product of inertia Pxy of the area of a rectangle about x- and y-axes coinciding with two adjacent sides of lengths a and b. The rectangle lies in the first quadrant.

2. Obtain the product of inertia for the area of the quarter circle about the x- and y-axes by direct integration (x- and y-axes passes through diameters).

$$\underline{\text{Ans.}}\quad Pxy = \frac{r^4}{8}$$

3. The moments of inertia of an area with respect to the principal axes of inertia x, y through a point P are $Ix = 32.0$ in.^{4} and $Iy = 12.0$ in^4. With the aid of Mohr's circle determine the moment of inertia Ix' and the product of inertia Px'y' for the area about axes x', y' through P and rotated 15 deg. clockwise from the axes x, y. <u>Ans.</u> $Ix' = 30.66$ in.^{4}, $Px'y' = -5$ in.^{4}

4. Determine the maximum and minimum moments of inertia about centroidal axes for the Z-section shown of previous problem no.6, and indicate the counterclockwise angle α made by the axis of maximum moment of inertia with the Xo-axis.

<u>Ans.</u> $\bar{I}$ max = 181.9 in.^{4}, I min = 20.7 in.^{4}, $\alpha = 30°8'$.

5. Determine the maximum and minimum moment of inertia about centroidal axes for the angle section of previous problem no.7 and indicate the counterclockwise angle α made by the axis of maximum moment of inertia with the Xo-axis.

II. MOMENTS OF INERTIA OF MASS:

The mass moment of inertia of a body is a measure of the inertial resistance to rotational acceleration. In Fig. C9 the body of mass m is caused to rotate about the axis O-O with an angular acceleration α. An element of mass dm has a component of acceleration tangent to its circular path equal to $r\alpha$, and the resultant tangential force on this element equals the force $r\alpha dm$. The moment of this force about the axis O-O is $r^2\alpha dm$. The sum of the moments of these forces for all elements is $r^2\alpha dm$. For a rigid body α is the same for all radial lines in the body and may be taken outside the integral sign. The remaining integral is known as the moment of inertia I of the mass m and is:

$$I = \int r^2 dm \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(C11)}$$

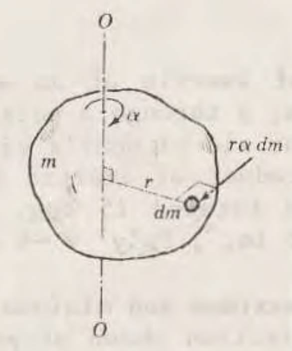


Fig. C9

This integral represents an important property of a body and is involved in the force analysis of any body which has rotational acceleration. Just as the mass m of a body is a measure of the resistance to translational acceleration, the moment of inertia is a measure of resistance to rotational acceleration due to the mass or inertia of the body.

If the mass density ρ is constant throughout the body, the moment of inertia becomes:

$$I = \rho\int r^2 dV.$$

where dV is the element of volume. In this case the integral by itself defines a purely geometrical property of the body. When the mass density is not constant but is expressed as a function of the coordinates of the body, it must be left within the integral sign and its effect accounted for in the integration process.

If the body is a wire or slender rod of length L and mass ρ per unit length, the moment of inertia about an axis becomes $I = \int r^2 \rho dL$, where r is the perpendicular distance from the element dL to the axis in question. If the body is a thin flat plate of area A and mass ρ per unit area, the moment of inertia is $I = \int r^2 \rho dA$. When ρ is constant over the plate, the expression becomes $I = \rho \int r^2 dA$. Thus the moment of inertia of the plate equals the mass per unit area times the "area" moment of inertia, described in Part I for axes in or normal to the plane of the area.

In general the coordinates which best fit the boundaries of the body should be used in the integration. It is particularly important to make a good choice of the element of volume dV. An element of lowest possible order should be chosen, and the correct expression for the moment of inertia of the element about the axis involved should be used. For example, in finding the moment of inertia of a right circular cone about its central axis, a cylindrical element in the form of a circular slice of infinitesimal thickness should be used. The differential moment of inertia for this element is the correct expression for the moment of inertia of a circular cylinder of infinitesimal thickness about its central axis.

The dimensions of mass moments of inertia are (mass) x $(\text{distance})^2$ and are usually expressed in the units $lb.ft.sec^2$. Frequently the units ft. slugs are used, where the slug is taken as the unit of mass.

Radius of Gyration: The radius of gyration "k" of a mass m about an axis for which the moment of inertia is I is:

$$k = \sqrt{\frac{I}{m}} \text{ or } I = k^2 m. \quad \ldots\ldots\ldots\ldots\ldots (C12)$$

Thus k is a measure of the distribution of mass of a given body about the axis in question, and its definition is analogous to the definition for the radius of gyration for second moments of area. If all the mass m could be concentrated at a distance k from the axis, the correct moment of inertia would be $k^2 m$. The moment of inertia of a body about a particular axis is frequently indicated by specifying the radius of gyration of the body about the axis and the weight of the body. The moment of inertia is then calculated from Eq. C12.

Transfer of Axes: If the moment of inertia of a body is known about a centroidal axis, it may be determined easily about any parallel axis. To prove this statement consider the two parallel axes in Fig. C10, one of which is a centroidal axis through the center of gravity G. The radial distances from the two axes to any element of mass dm are r_o and r, and the separation of the axes is d. Substituting the law of cosines $r^2 = r_o^2 + d^2 + 2r_o d \cos\theta$ into the definition for the moment of inertia about the noncentroidal axis gives:

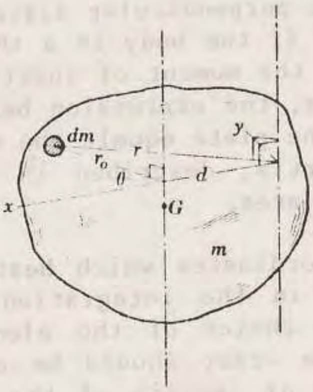


Fig. C10

$$I = \quad r^2 dm = \quad (ro^2 + d^2 + 2rod \cos 0)\ dm,$$
$$= \quad ro^2 dm + d^2 \quad dm + 2d \quad ydm.$$

The first integral is the moment of inertia $\bar{I}$ about the centroidal axis, the second integral is md^2, and the third integral equals zero since the y-coordinate of the center of gravity with respect to an origin at G is zero. Thus the parallel-axis theorem is:

$$I = \bar{I} + md^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (C13)$$

It must be remembered that the transfer cannot be made unless one axis passes through the center of gravity and unless the axes are parallel. When the expression for the radii of gyration are substituted in Eq. C13, there results which is the parallel axis theorem for

$$k^2 = \bar{k}^2 + d^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (C13a)$$

obtaining the radius of gyration k about an axis a distance d from a parallel centroidal axis for which the radius of gyration is $\bar{k}$.

Product of Inertia: In a few problems of advanced mechanics the integrals: $Ixy = \int xydm$; $Iyz = \int yzdm$; $Ixz = \int xzdm$; are useful. These integrals are called the products of inertia of the mass m. They may be either positive or negative. In general, a three-dimensional body has three moments of inertia about the three mutually perpendicular coordinate axes and three products of inertia about the three coordinate planes. For an unsymmetrical body of any shape it is found that for a given origin of coordinates there is one orientation of axes for which the products of inertia vanish. These axes are called the "Principal Axes of Inertia". The corresponding moments of inertia about these axes are known as the "Principal Moments of Inertia" and include the maximum possible value, the minimum possible value, and an intermediate value for any orientation of axes about the given origin.

Moment of Inertia with Respect to a Plane: The moment of inertia of a body with respect to a plane is useful in some problems primarily as an aid to the calculation of the moment of inertia with respect to a line. The moment of inertia with respect to the y-z plane is defined as x^2dm and that with respect to the x-z plane is y^2 dm. Since $x^2 + y^2 = r^2$, where r is the distance from dm to the z-axis, the moment of inertia Iz about the z-axis is:

$$Iz = \int r^2 dm = \int x^2 dm + \int y^2 dm.$$

Similar expressions may be written for the two other axes.

SAMPLE PROBLEMS

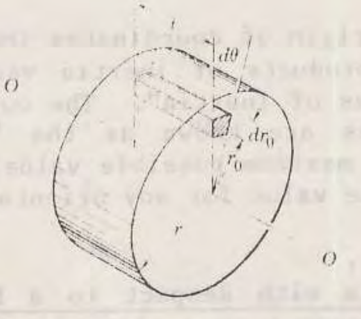
1. Determine the moment of inertia and radius of gyration of a homogeneous right circular cylinder of mass m and radius r about its central axis O-O.

Solution: An element of mass in cylindrical coordinates is $dm = \rho dV = \rho t r_0 dr_0 d\theta$. The moment of inertia about the axis of the cylinder is:

$$I = \int r^2{}_0 dm = \rho t \int_0^{2\pi} \int_0^{r} r^3{}_0 dr_0 \; \theta = \rho t \frac{\pi r^4}{} \; \frac{1}{2} mr^2 \qquad \underline{\text{Ans}}.$$

The radius of gyration is $k = \sqrt{\frac{I}{m}} = \frac{r}{\sqrt{2}}$

The result $I = \frac{1}{2} mr^2$ applies only to a solid homogeneous circular cylinder and cannot be used for any other wheel of circular periphery.



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2. Determine the moment of inertia and radius of gyration of a homogeneous solid sphere of mass m and radius r about a diameter.

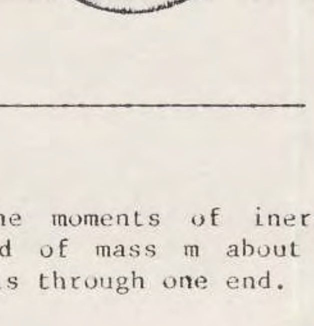
Solution: A circular slice of radius y and thickness dx is chosen as the volume element. From the result of prob. 1, the moment of inertia about the x-axis of the elemental cylinder is:

$$dIx = \frac{1}{2}(dm)\ y^2 = \frac{1}{2}(\pi\rho y^2 dx)\ y^2 = \frac{\pi\rho}{2}(r^2 - x^2)^2\ dx,$$

where ρ is the constant mass density of the sphere. The total moment of inertia about the x-axis is:

$$Ix = \frac{\pi\rho}{2}\int_{-r}^{+r}(r^2 - x^2)^2\ dx = \frac{8}{15}\pi\rho r^5 = \frac{2}{5}mr^2 \qquad \text{Ans.}$$

The radius of gyration is: $k = \sqrt{\frac{I}{m}} = \sqrt{\frac{2}{5}}\,r$ Ans.



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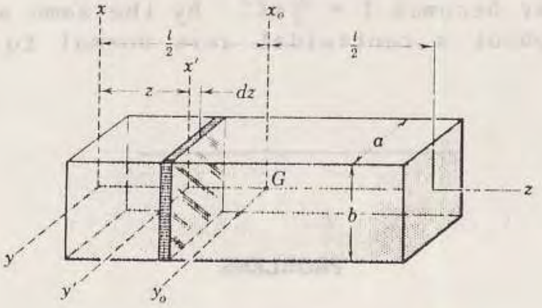
3. Determine the moments of inertia of the homogeneous rectangular parallelepiped of mass m about the centroidal x_0 - and z-axes and about the x-axis through one end.

Solution: A transverse slice of thickness dz is selected as the element of volume. The moment of inertia of this slice of infinitesimal thickness equals the moment of inertia of the area of the section times the mass per unit area ρdz. Thus the moment of inertia of the transverse slice about the y'-axis is:

$$dI_{y'} = (\rho dz)\left(\frac{1}{12}ab^3\right),$$

and that about the x'-axis is:

$$dI_{x'} = (\rho dz)\left(\frac{1}{12}a^3 b\right).$$



As long as the element is a plate of differential thickness, the principle of Eq. C3 may be applied to give:

$$dI_z = dI_{x'} + dI_{y'} = (\rho dz)\frac{ab}{12}(a^2+b^2).$$

These expressions may now be integrated to obtain the desired results.

The moment of inertia about the z-axis is:

$$I_z = \int dI_z = \frac{\rho ab}{12}(a^2+b^2)\int_0^{\ell} dz = \frac{1}{12}m(a^2+b^2) \qquad \underline{\text{Ans.}}$$

where m is the mass of the block. By interchanging symbols the moment of inertia about the x_0-axis is:

$$I_{x_0} = \frac{1}{12}m(a^2+\ell^2) \qquad \underline{\text{Ans.}}$$

The moment of inertia about the x-axis may be found by the parallel-axis theorem, Eq. C13. Thus:

$$I_x = I_{x_0} + m\left(\frac{\ell}{2}\right)^2 = \frac{1}{12}m(a^2+4\ell^2) \qquad \underline{\text{Ans.}}$$

The last result may be obtained by expressing the moment of inertia of the elemental slice about the x-axis and integrating the expression over the length of the bar. Again by the parallel-axis theorem:

$$dIx = dIx' + z^2 dm = (\rho\, dz)\,(\frac{1}{12}\, a^3 b) + z^2 \rho\, ab dz,$$
$$= \rho ab\ (\frac{a^2}{12} + z^2)\ dz.$$

Integrating gives the result obtained previously:

$$Ix = \rho ab \int_0^{\ell} (\frac{a^2}{12} + z^2)\ dz = \frac{\rho ab \ell}{3}\ (\ell^2 + \frac{a^2}{4}) = \frac{1}{12}\, m\ (a^2 + 4\ell^2).$$

The expression for Ix may be simplified for a long prismatical bar or slender rod whose transverse dimensions are small compared with the length. In this case a^2 may be neglected compared with $4\ell^2$, and the moment of inertia of such a slender bar about an axis through one end normal to the bar becomes $I = \frac{1}{3} m\ell^2$. By the same approximation the moment of inertia about a centroidal axis normal to the bar is $I = \frac{1}{12}\, m\ell^2$.

PROBLEMS

1. A bar 10 in long has a square cross-section 1 in. on a side. Determine the percentage error e in using the approximate formula $I = \frac{1}{3} m\ell^2$ for the moment of inertia about an axis normal to the bar and through the center of one end parallel to an edge (see sample problem no.3 above). Ans. $e = 0.249\%$

2. Determine the moment of inertia of a circular ring of mass m and inside and outside radii r_1 and r_2, respectively, about its central polar axis.

Ans. $I = \frac{1}{2}\, m\ (r_2^2 + r_1^2)$.

3. Calculate the moment of inertia of a homogeneous right circular cone of mass m and base radius r about the cone axis.

Ans. $I = \frac{3}{10}\, mr^2$.

4. The moment of inertia of a body with respect to the x-y plane is 0.202 lb.ft.sec.^{2}, and that with respect to the y-z plane is 0.440 lb.ft.sec.^{2} The radius of gyration about the y-axis is 1.20 ft. Find the weight W of the body.

5. Determine the moment of inertia of the elliptical cylinder of mass m about the cylinder axis 0-0.

<u>Ans</u>. $I = \frac{1}{4} m (a^2 + b^2)$.

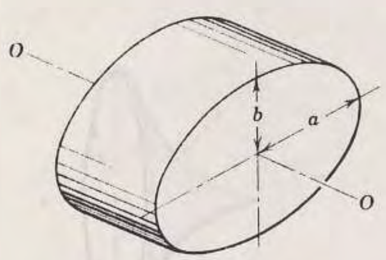


Fig. 5

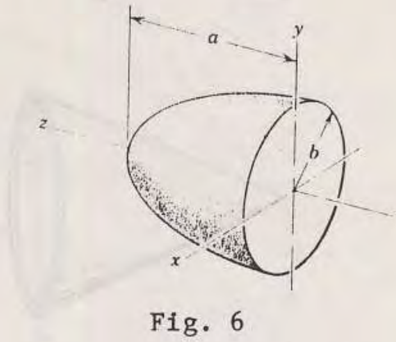


Fig. 6

6. Determine the moment of inertia about the z-axis of the homogeneous solid paraboloid of revolution of mass m.

7. Determine the moments of inertia of the homogeneous right circular cylinder of mass m about the x_0-, x-, and y'-axes shown.

<u>Ans</u>. $I_{x0} = \frac{1}{12} m (3r^2 + \ell^2)$, $I_x = \frac{1}{12} m (3r^2 + 4\ell^2)$, $Iy = \frac{3}{2} mr^2$.

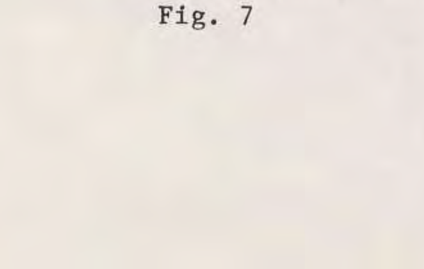


Fig. 7

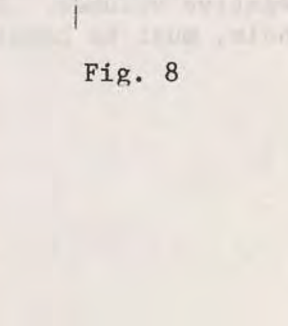


Fig. 8

8. Determine the moments of inertia of the half spherical shell with respect to the x- and z-axes. The mass of the shell is m, and its thickness is negligible compared with the radius r.

$$\underline{\text{Ans}}.\quad I_x = I_z = \frac{2}{3}\, mr^2.$$

9. Determine the moment of inertia of the conical shell of mass m about the axis of rotation. Wall thickness is negligible.

$$\underline{\text{Ans}}.\quad I_z = \frac{1}{2}\, mr^2.$$

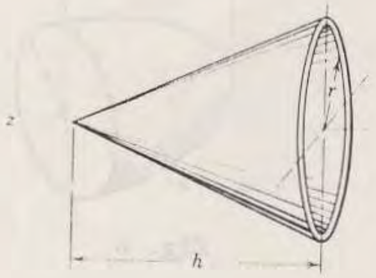


Fig. 9

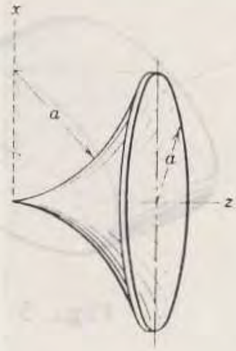


Fig. 10

10. Determine the moment of inertia about the z-axis of the bell-shaped shell of uniform small thickness if the mass is m.

$$\underline{\text{Ans}}.\quad I_z = \frac{15\pi - 44}{6(\pi - 2)}\, ma^2.$$

Composite Bodies: The defining integral, Eq. C11, involves the square of the distance from the axis to the element and so is always positive. Thus, as in the case of area moments of inertia, the mass moment of inertia of a composite body is the sum of the moments of inertia of the individual parts about the same axis. It is often convenient to consider a composite body as defined by positive volumes and negative volumes. The moment of inertia of negative element, such as a hole, must be considered a minus quantity.

PROBLEMS

1. Calculate the moment of inertia about the z-axis of the cylinder with the hemispherical cavity if the net mass is m.

Ans. $I_z = \frac{7}{10} ma^2$.

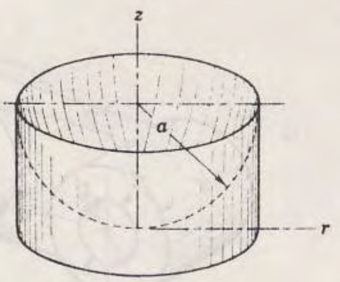


Fig. 1

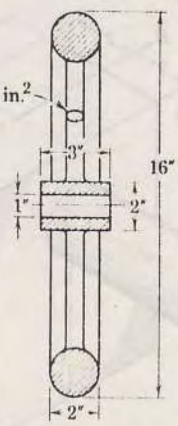


Fig. 2

2. Calculate the moment of inertia of the steel handwheel about its axis. There are six spokes, each of which has a uniform cross sectional area of 0.40 $in.^2$

Ans. $I = 0.431$ $lb.ft.sec.^2$

3. Determine the radius of gyration of the homogeneous rotor, shown in section, about its central axis. Ans. $k = 2.43$ in.

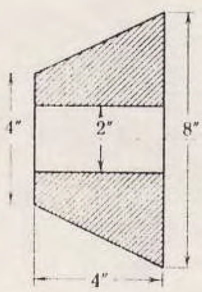


Fig. 3

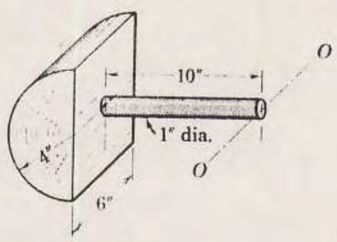


Fig. 4

4. Determine the moment of enertia of the mallet with respect to the axis O-O. The head is made from hardwood weighing 65 lb./ft.^{3}, and the handle is made from steel weighing 0.283 lb./in.^{3}

Ans. Io = 0.1896 lb.ft.sec.^{2}

5. Determine the moments of inertia of the steel body shown about axes A and B.

Ans. I_A = 0.1446 lb.ft.sec.^{2}; I_B = 0.302 lb.ft.sec.^{2}

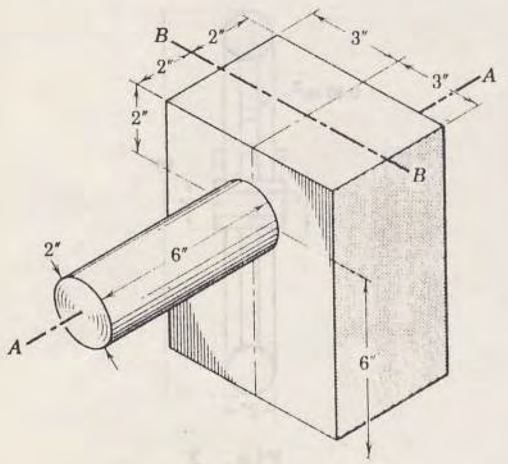


Fig. 5

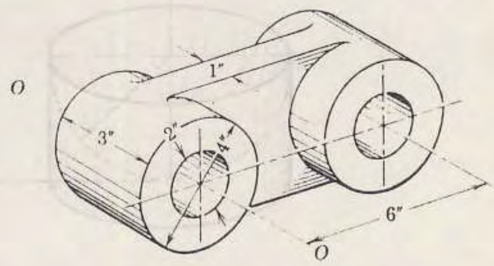


Fig. 6

6. Determine the radius of gyration of the symmetrical steel link about the axis O-O. Ans. k = 4.36 in.